# Homework 6

**MATH 472**

1) Please email me your homework as a single pdf file.
2) Show your work clearly. Justify all your answers.

1. (**Approximation**) Let $a$, $b$, and $c$ be positive numbers. The set of points $(x, y, z)$ in $\mathbb{R}^3$ such that

$$(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$$

is called a hyperboloid. Find the equation of the tangent plane to this hyperboloid at $(x_0, y_0, z_0)$ with $z_0$ positive.

2. (**Implicit Function Theorem**) Check directly where we can solve the equation $F(x, y) = y^2 + y + 3x + 1 = 0$ for $y$ in terms of $x$. Check that your answer agrees with the answer you expect from the Implicit Function Theorem. Compute $dy/dx$.

3. Recall the following theorem discussed in class.

**Theorem 1** *Let $f : D \to \mathbb{R}$ where $D$ is a rectangle. The function $f$ is integrable if and only if $\{P_n\}$ is a sequence of partitions of $D$ such that* $\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f)$. *Moreover,*

$$\iint_D f = \lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f)$$

Let $D = [0, 1] \times [0, 1]$. Define

$$f(x, y) = \begin{cases} 5 & \text{if } (x, y) \in D \text{ and } x > 1/2 \\ 1 & \text{if } (x, y) \in D \text{ and } x \le 1/2. \end{cases}$$

Using the theorem above show that $f$ is integrable on $D$. Using the function $f$ conclude that an integrable function need not be continuous.

4. (Fubini's Theorem) Let $D = [0, 1] \times [0, 1]$. Define

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 2y & \text{if } x \text{ is irrational.} \end{cases}$$

Show that $\int_0^1 \int_0^1 f(x,y) dydx = 1$.

Let $\{P_{2k}\} = \{0, \frac{1}{2k}, \ldots, \frac{k-1}{2k}, \frac{1}{2}, \frac{k+1}{2k}, \ldots, \frac{2k-1}{2k}, 1\}$ be a partition of $[0,1]$ into $2k$ subintervals. Now show that $\iint_D f(x,y)$ does not exist. Try to do this by using the theorem stated above and considering the sequence of partitions of $D$ created by $\{P_{2k}\}$. Then show that $U(P_{2k}, f) \to \frac{5}{4}$ and $L(P_{2k}, f) \to \frac{3}{4}$ as $k \to \infty$.

(This shows that Fubini's Theorem does not hold in this case.)

5. (Fubini's Theorem) Let $D = [0,1] \times [0,1]$. Define

$$f(x,y) = \begin{cases} \frac{(x-y)}{(x+y)^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that $\int_0^1 \int_0^1 \frac{(x-y)}{(x+y)^3} dxdy = -\frac{1}{2}$ and $\int_0^1 \int_0^1 \frac{(x-y)}{(x+y)^3} dydx = \frac{1}{2}$. Use this result to conclude that $f$ is not integrable on $D$.

(Hint: Consider writing $\frac{(x-y)}{(x+y)^3} = \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}$.)

6. Verify that if $f(x,y)$ is continuous and $[a,b]$ is an interval, then

$$\int_a^b \int_a^x f(x,y)\ dy\ dx = \int_a^b \int_y^b f(x,y)\ dx\ dy.$$

7. Let $f(x,y) = e^{y^2}$, where $D$ is the region between the curves $y = 2x$, $x = 0$, and $y = 2$. Compute $\iint_D f$.

8. Reverse the order of integration and evaluate the resulting integral from

$$\int_0^2 \int_{x^2}^{2x} (2x+3)\ dy\ dx.$$