Homework 6

MATH 471

All work must be shown clearly. You must justify all your answers. (Students taking the course through Engineering Outreach may email me your solutions in a pdf file.)

1. Consider the partition $P = \{0, 1/4, 1/2, 1\}$ of the interval $[0, 1]$. Compute $L(P, f)$ and $U(P, f)$ for the function $f(x) = -x^2,\ x \in [0, 1]$.

2. Suppose that a function $f : [a, b] \to \mathbb{R}$ is bounded, and let P and Q be two partitions of $[a, b]$ such that $P \subseteq Q$, that is, Q is a refinement of P. Show that $U(Q, f) \leq U(P, f)$.
   (Note: In one of the lectures it was proved that $L(P, f) \leq L(Q, f)$ and you may want to use that proof as a guide. Together, this result is often called the *Refinement Lemma*.)

3. If a function $f : [-2, 3] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 2|x| + 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational}, \end{cases}$$

   prove that f is not Riemann integrable.

4. Give an example of a function $f : [0, 1] \to \mathbb{R}$ such that

   (a) f is bounded but f is not Riemann integrable.

   (b) f is Riemann integrable but not monotone.

   (c) f is Riemann integrable but neither continuous nor monotone.

   (Your examples in (b) and (c) should be different.)