Additional Problems for Midterm I

MATH 471

1. Show that the sequence $\{\frac{n}{n^2-3}\}_{n=1}^{\infty}$ converges.

2. If a sequence $\{a_n\}$ converges to 0, and a sequence $\{b_n\}$ is bounded, then show that the sequence $\{a_n b_n\}$ converges to 0. Give an example to show that this is not true when $\{b_n\}$ is not bounded.

3. Use the monotone convergence theorem to show that the given sequence is convergent. Find the limit.
$$\left\{\frac{1}{1+n^2}\right\}_{n=1}^{\infty}.$$
(Ans. limit $= 0$.)

4. Show that for a real c such that $|c| < 1$,
$$\lim_{n\to\infty} c^n = 0.$$
What is the nature of convergence for $c = 1$ and $|c| > 1$?
(See class notes: lectures 6 & 7.)

5. Define
$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0 \end{cases}$$
At what points is the function $f : \mathbb{R} \to \mathbb{R}$ continuous? Justify your answer.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $g(x) = f(x^2 + x^3)$ is continuous.
Note: For 5. & 6. you can assume that polynomials are continuous.

7. State the Intermediate Value Theorem. Using the IVT prove that every polynomial of odd degree has at least one real root. (See lecture #12)

8. In class we proved the Extreme Value Theorem by showing that every continuous function on a closed and bounded set attains a maximum. Modify the proof to show that every continuous function on a closed and bounded set attains a minimum.

9. Let $f(x) = |x|$. Show that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.
(Try to show that f is Lipschitz.)

10. Prove that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous, then so is the sum $f + g$.

11. Find the following limits or determine that they do not exist:
    (a) $\lim_{x\to 0, x>0} \frac{x+\sqrt{x}}{2+\sqrt{x}}$
    (Ans. 0)
    (b) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$
    (Ans. 1/2)
    (c) $\lim_{x\to 0} \frac{1+\frac{1}{x^2}}{1+\frac{1}{x}}$
    (Ans. Does not exist)