1. (15 pts) Evaluate the following series, or state that it diverges.

(a) $\sum_{n=2}^{\infty}(-1)^n$

diverges ; geometric series with $r = -1$
$|r| = 1$

(b) $\sum_{n=1}^{\infty}\frac{5^n}{3^n}$

diverges ; geometric series with $r = 5/3 > 1$

The other version had $\sum \frac{4^n}{3^n}$ which is similar

(c) $\sum_{n=0}^{\infty}\frac{1}{2^n}$ $\quad r = 1/2 < 1$

$$= \frac{1}{1 - 1/2} = \boxed{2}$$

2. (5 pts) Does the series $\sum_{n=1}^{\infty}\frac{2n}{5n+1}$ converge or diverge? Explain.

$$\lim_{n\to\infty}\frac{2n}{5n+1} = 2/5 \neq 0$$

The series diverges by the Divergence Test.

The other version got $\sum_{n=1}^{\infty}\frac{2n}{3n+1}$; $\lim \frac{2n}{3n+1} = 2/3 \neq 0$

3. (12 pts) Use the Integral Test to determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges. First show that the conditions for using the Integral Test are met. Use proper notation.

Let $f(x) = \dfrac{1}{x \ln x}$. Then $a_n = \dfrac{1}{n \ln n} = f(n)$.

f is continuous, decreasing and positive on $[2, \infty)$.

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x \ln x}\, dx \qquad \boxed{\begin{aligned} u &= \ln x \\ du &= \frac{dx}{x} \end{aligned}}$$

$$= \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{du}{u}$$

$$= \lim_{b \to \infty} \Big[\ln u\Big]_{\ln 2}^{\ln b} = \lim_{b \to \infty} \Big[\ln(\ln b) - \ln(\ln 2)\Big]$$

$$= \infty$$

Thus, by the Integral Test, the given series diverges.

4. (10 pts each) Determine whether the following series are convergent or divergent. State the test used, apply the test, and state the result. Show all work and use proper notation.

(a)

$$\sum_{n=1}^{\infty} \left(\frac{n^2+1}{n^3+1}\right)^n$$

Root test: $\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+1}{n^3+1}\right)^n} = \lim_{n\to\infty} \frac{n^2+1}{n^3+1} = 0 < 1$

The series converges.

(b)

$$\sum_{n=1}^{\infty} \frac{n!}{5^n}$$

Ratio test: $\lim_{n\to\infty} \frac{(n+1)!\, 5^n}{5^{n+1}\; n!} = \lim_{n\to\infty} \frac{n+1}{5} = \infty > 1$

The series diverges

(c)

$$\sum_{n=1}^{\infty} \frac{1}{3^n+1}$$

Comparison Test : compare with $\sum \frac{1}{3^n}$

$\sum \frac{1}{3^n}$ is a geometric series with $r = \frac{1}{3} < 1$

and so converges.

$$\frac{1}{3^n+1} < \frac{1}{3^n} \implies \sum \frac{1}{3^n+1} \text{ also converges}$$

The other version had $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ which follows a similar argument

5. (12 pts) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n-1}}$ is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n-1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n-1}}$$

. Let's compare with $\sum \frac{1}{\sqrt{n}}$ which diverges as a p-series with $p = 1/2$.

Comparison Test: $\frac{1}{\sqrt{n-1}} > \frac{1}{\sqrt{n}}$

and so $\sum \frac{1}{\sqrt{n-1}}$ also diverges. Thus the given series is not absolutely convergent.

Alt. series test: $\sum \frac{(-1)^n}{\sqrt{n-1}}$

(i) $a_n = \frac{1}{\sqrt{n-1}} \to 0$ as $n \to \infty$

(ii) $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n+1}}$

By the alternating series test, we have convergence.

Therefore, the given series converges conditionally.

6. (4 pts) Circle the correct statement(s).

(A) The geometric series $\sum r^n$ converges if $|r| > 1$.

(B) A series $\sum a_n$ diverges if $\lim_{n\to\infty} a_n = 1$.

(C) A positive series $\sum a_n$ diverges by the Ratio Test if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$.

(D) A positive series $\sum a_n$ converges by the Ratio Test if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

7. (10 pts) Find the 3$^{\text{rd}}$ Taylor polynomial p_3 of the funtion $\cos x$ centered at $x = 0$.

$f(x) = \cos x$ $\qquad c_0 = f(0) = \cos(0) = 1$

$f'(x) = -\sin x$ $\qquad c_1 = f'(0) = -\sin(0) = 0$

$f''(x) = -\cos x$ $\qquad c_2 = \frac{f''(0)}{2} = \frac{-\cos(0)}{2} = -1/2$

$f'''(x) = \sin x$ $\qquad c_3 = \frac{f'''(0)}{3!} = \frac{\sin(0)}{3!} = 0$

$p_3 = 1 - \frac{1}{2}x^2$

8. (12 pts) Find the interval and the radius of convergence of

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n3^n}.$$

(Make sure to check the end-points of the interval separately!)

Ratio Test for absolute convergence

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)3^{n+1}}\frac{n\,3^n}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\frac{n}{(n+1)}\right| = \frac{|x|}{3} < 1$$

$$\Rightarrow \quad |x| < 3$$

$$\Rightarrow \quad -3 < x < 3$$

Test end-points :

$x = 3$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{n\,3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$: alternating series test

(a) $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$ ✓

(b) decreasing in size? $\displaystyle\frac{1}{n+1} < \frac{1}{n}$ ✓

By the alt. series test there is convergence at $x = 3$.

$x = -3$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(-3)^n}{n\,3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n 3^n}{n\,3^n} = \sum_{n=1}^{\infty}\frac{(-1)^{\overbrace{2n}^{\to\text{even}}}}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$

which is the harmonic series and diverges.

Therefore, interval of convergence is $(-3, 3]$.

Radius $= 3$