1. (10 pts) Find the Maclaurin's series for $\sin(x^2)$.
Your final answer should be in Σ form.
(Recall that the Maclaurin's series for $\sin(x)$ is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$)

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

2. (10 pts) Use the series found in 1. above to evaluate $\int_0^1 \sin(x^2)\, dx$.
Your final answer should be in Σ form.

$$\int_0^1 \sin(x^2)\, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{4n+2}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left. \frac{x^{4n+3}}{4n+3} \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{4n+3}$$

3. (10 pts) Find the Taylor series for the function $f(x) = e^{4x}$ about the point $a = 1/4$. Your final answer should be in Σ form.

$$f(x) = e^{4x}$$

$$f'(x) = 4e^{4x}$$

$$f''(x) = 4^2 e^{4x}$$

$$\vdots$$

$$f^{(k)}(x) = 4^k e^{4x} \Rightarrow f^{(k)}(1/4) = 4^k e$$

$$C_k = \frac{f^{(k)}(\frac{1}{4})}{k!} = \frac{4^k e}{k!}$$

$$\boxed{\text{Taylor series of } e^{4x} = \sum_{k=0}^{\infty} \frac{4^k e}{k!}\left(x - \frac{1}{4}\right)^k}$$

4. (10 pts) Use the Maclaurin's series of e^x to find

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{x^2/2! + x^3/3! + \cdots}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{2!} + \frac{x}{3!} + \underbrace{\cdots}_{\text{higher powers of } x}$$

$$= \frac{1}{2}$$

5. (10 points)
Using the fact that $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$, find the Maclaurin's series for $\frac{1}{9+x^2}$.
What is the radius of convergence?

$$\frac{1}{9+x^2} = \frac{1}{9\left(1+\frac{x^2}{9}\right)} = \frac{1}{9} \frac{1}{1-\left(-\frac{x^2}{9}\right)}$$

$$= \frac{1}{9} \sum_{k=0}^{\infty} \left(-\frac{x^2}{9}\right)^k = \frac{1}{9} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{9^k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{9^{k+1}}$$

Converges when $\left|-\frac{x^2}{9}\right| < 1$

or, $\left|\frac{x^2}{9}\right| < 1$

or, $|x^2| < 9$

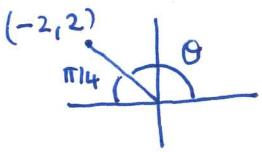
or $|x| < 3$

or $-3 < x < 3$

Radius $= 3$

6. (10 points)

(a) Express the Cartesian coordinates $(-2, 2)$ in polar coordinates.



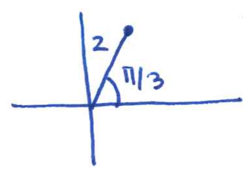
$$x = -2, \quad y = 2 \quad \Rightarrow \quad r = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \pi - \pi/4 = 3\pi/4$$

$$(r, \theta) = (2\sqrt{2}, 3\pi/4)$$

(b) Express the polar coordinates $(2, \pi/3)$ in Cartesian coordinates.

$$r = 2, \quad \theta = \pi/3$$



$$x = 2\cos \pi/3 = 1$$

$$y = 2\sin \pi/3 = \sqrt{3}$$

$$(x, y) = (1, \sqrt{3})$$

7. (10 points) Eliminate t from the parametric curve:

$$\left.\begin{array}{l} x = 1 + \sin t \\ y = 2 + \cos t \end{array}\right\}, \quad 0 \le t \le \pi/2.$$

to give the equation in Cartesian form.
Sketch the graph and make sure to indicate the direction by arrows.

$$x - 1 = \sin t$$

$$y - 2 = \cos t$$

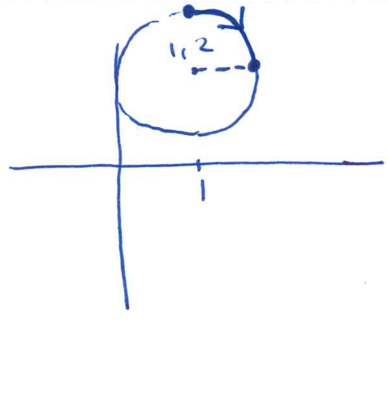
$$\sin^2 t + \cos^2 t = 1$$

$$\Rightarrow \quad (x-1)^2 + (y-2)^2 = 1$$

Circle with center $(1, 2)$ and radius 1.

$t = 0 \quad x = 1 \quad y = 3$

$t = \pi/2 \quad x = 2 \quad y = 2$



$0 \le t \le \pi/2$ is the quarter of the circle

8. (10 pts) Convert the polar equation $r = 3\sin\theta$ to Cartesian form. Describe the resulting curve.

$$r = 3\sin\theta$$

$$r^2 = 3r\sin\theta$$

$$x^2 + y^2 = 3y$$

$$x^2 + y^2 - 3y = 0$$

$$x^2 + y^2 - 3y + \frac{9}{4} = 9/4$$

$$x^2 + (y - 3/2)^2 = \left(\frac{3}{2}\right)^2$$

circle with center $(0, 3/2)$, radius $= \frac{3}{2}$

9. (10 pts) For the polar equation given in 8. find the slope of the tangent line at $\theta = \pi/3$.

$f(\theta) = 3\sin\theta$

$f'(\theta) = 3\cos\theta$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

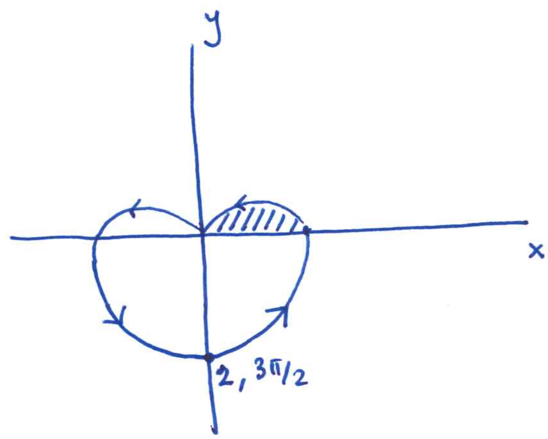
$$= \frac{3\cos\theta\sin\theta + 3\sin\theta\cos\theta}{3\cos\theta\cos\theta - 3\sin\theta\sin\theta}$$

$$= \frac{2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta}$$

$$\theta = \pi/3 \Rightarrow \text{slope} = \frac{2\cos\pi/3\,\sin\pi/3}{\cos^2(\pi/3) - \sin^2(\pi/3)} = \frac{\cancel{2}\,\frac{1}{\cancel{2}}\,\sqrt{3}/2}{\frac{1}{4} - 3/4}$$

$$= \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$$

10. (10 pts) Find the area of the region bounded by the curve $r = 1 - \sin\theta$, the x-axis, and the y-axis in the first quadrant.



Area of the shaded region

$$= \int_0^{\pi/2} \frac{(1-\sin\theta)^2}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 - 2\sin\theta + \sin^2\theta\right) d\theta \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[1 - 2\sin\theta + \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 - 2\sin\theta + \frac{1}{2} - \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{3}{2} - 2\sin\theta - \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\cos\theta - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{3}{2}\,\frac{\pi}{2} + 2\underbrace{\cos \pi/2}_{\to 0} - \frac{\overbrace{\sin \pi}^{\to 0}}{4} - \left(\frac{3}{2}\cdot 0 + 2\cos 0 - \frac{\sin 0}{4}\right)\right]$$

$$= \frac{1}{2}\left(\frac{3}{4}\pi - 2\right) = \frac{3}{8}\pi - 1$$