MATH 420: Practice Problems

1. Describe the set of points given by $|z+i| \leq 3$.

2. Find $(\sqrt{3}+i)^7$. Give your answer in rectangular form (i.e. as $x+iy$). (Ans. $-64(\sqrt{3}+i)$)

3. Find
(a) $(\sqrt{3}+i)^{1/2}$
(b) $8^{1/6}$
In each case indicate the principal value.
(Ans. $\pm\sqrt{2}$, $\pm\frac{1+i\sqrt{3}}{\sqrt{2}}$, $\pm\frac{1-i\sqrt{3}}{\sqrt{2}}$)

4. Show that
(a) $\text{Log}(i^5) = \text{Log}(i) = i\pi/2$.
(b) $5\text{Log}(i) = i5\pi/2$
Hence $\text{Log}(i^5) \neq 5\text{Log}(i)$.
(Here Log is the principal value of the log function.)

5. Evaluate the integral
$$\int_C \overline{z}\ \mathrm{d}z$$
when C is the right-hand half
$$z = 2e^{i\theta}; \quad -\pi/2 < \theta < \pi/2$$
of the circle $|z| = 2$.

6. Using the Fundamental Theorem of Algebra prove that every polynomial equation $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ has *exactly* n roots.

7. Using Gauss Mean Value Theorem, find
$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2(\pi/6 + 2e^{i\theta})\ \mathrm{d}\theta.$$
(Ans. 1/4)

8. Find the Laurent series about the indicated singularity. Identify the type of singularity:
(a) $(z-3)\sin\frac{1}{z+2}$; $z=-2$ (Ans. essential singularity)
(c) $\frac{z-\sin z}{z^3}$; $z=0$ (Ans. removeable singularity)

9. Evaluate

$$\int_C \frac{\mathrm{d}z}{z^3(z+4)}$$

taken counterclockwise around the circle C (a) $|z| = 2$, (b) $|z+2| = 3$. (Ans. (a) $\frac{\pi i}{32}$, (b) 0)

10. Using the method of residues, evaluate

$$\int_0^\infty \frac{x \sin 2x}{x^2+3} \, \mathrm{d}x$$

(Ans. $\frac{\pi}{2} e^{-2\sqrt{3}}$)

11. Show that

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a \sin\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad (-1 < a < 1)$$

12. Find a harmonic conjugate of the harmonic function $u(x, y) = x^3 - 3xy^2$.

13. The transformation $w = e^z$ maps the horizonal strip $0 < y < \pi$ onto the upper half plane $v > 0$. The function

$$h(u, v) = \mathrm{Re}(w^2) = u^2 - v^2$$

is harmonic in that half plane. With the aid of a theorem done in class show that the function $H(x, y) = e^{2x} \cos 2y$ is harmonic in the strip. Verify this result directly.

14. Practice all problems from HW 1-7

15. Be familiar with all examples solved in class