University of Idaho

MATH 430
Advanced Linear Algebra

Session 10

Range and Null space of a linear transformation

V - vector space with basis $\{v_1, v_2, \dots, v_m\}$.

$T : V \longrightarrow W$ Let T be <u>linear.</u>

Let $x \in V$. There are scalars $a_1, \dots, a_m$ s.t.

$$x = a_1 v_1 + a_2 v_2 + \dots + a_m v_m.$$

$$T(x) = T(a_1 v_1 + a_2 v_2 + \dots + a_m v_m)$$

$$= a_1 T(v_1) + a_2 T(v_2) + \dots + a_m T(v_m)$$

T is <u>completely</u> determined by its action on the basis vectors $v_1, v_2, \dots, v_m$.

Knowing $T(v_1) \dots T(v_m)$ gives us $T(x)$.

Theorem: V & W are vector spaces and $T: V \to W$ is a linear transformation. If $B = \{v_1, \dots, v_m\}$ is basis of V, then

$$R(T) = \text{span}\{T(v_1), T(v_2), \dots, T(v_m)\}.$$

Example: $T : \underbrace{P_2(\mathbb{R})}_{V} \longrightarrow \underbrace{M_{2\times 2}(\mathbb{R})}_{W}$

$$T(f(x)) = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}$$

$\dim(P_2) = 3$

say $f(x) = 1 + x^2 \in P_2$

$f(1) = 1 + 1 = 2$

$f(2) = 1 + 4 = 5$

Find $R(T)$ and $N(T)$.

$B = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$

$$T(1) = \begin{bmatrix} 1-1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad T(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} 1^2 - 2^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R(T) = \text{span}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{Basis of } R(T)} \right\}$$

$\dim(R(T)) = 2$

$f = a_0 + a_1 x + a_2 x^2$

$T(f) = a_0 T(1) + a_1 T(x) + a_2 T(x^2)$

$N(T)$ = all polynomials in P_2 that are mapped to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$f(x) \longmapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$f(0) = a_0, \quad f(1) = a_0 + a_1 + a_2, \quad f(2) = a_0 + 2a_1 + 4a_2$$

$$f(1) = f(2) \implies a_1 + a_2 = 2a_1 + 4a_2$$

$$\implies a_1 = -3a_2$$

$$f(0) = 0 \implies a_0 = 0$$

$f(x)$ is in $N(T)$ if

$$f(x) = -3a_2 x + a_2 x^2$$

$$= a_2(-3x + x^2) \qquad a_2 \in \mathbb{R}$$

$$N(T) = \text{Span}\{\underbrace{-3x + x^2}_{\text{Basis of } N(T)}\}$$

$$\dim(N(T)) = 1$$

Nullity$(T) = \dim(N(T))$, Rank of $(T) = \dim(R(T))$

In the above example $V = P_2(\mathbb{R})$

$$\dim(V) = 3 = \underbrace{\dim(R(T))}_{=2} + \underbrace{\text{nullity}(T)}_{=1}$$

<u>Dimension Theorem</u> : Let V and W be vector spaces and let $T: V \longrightarrow W$ be linear. If V is finite dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Proof of dimension theorem :

Let $\dim(V) = n$ and $\text{nullity}(T) = k$.
Want to show: $\dim(R(T)) = n - k$.

Let $\{v_1, \dots, v_k\}$ be a basis of $N(T)$.
Since $N(T) \subseteq V$, we can expand this to a basis of V.

Let $B = \{ v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n \}$ be a basis of V.

$v_1 \mapsto 0$, $v_2 \mapsto 0$, $v_k \mapsto 0$

<u>Claim</u>: $S = \{ T(v_{k+1}), T(v_{k+2}), \dots, T(v_n) \}$ is a basis of $R(T)$.

(i) S spans $R(T)$: We know that

$$R(T) = \text{Span}\{T(v_1), \dots, T(v_k), T(v_{k+1}), \dots, T(v_n)\}$$
$$= \text{span}\{T(v_{k+1}), \dots, T(v_n)\} = \text{span}(S)$$

(ii) S is l.i. : Let

$$b_{k+1} T(v_{k+1}) + b_{k+2} T(v_{k+2}) + \cdots + b_n T(v_n) = 0$$

$$\cancel{\Rightarrow T \; b_{k+1} T(v}$$

$$\Rightarrow T(b_{k+1} v_{k+1} + b_{k+2} v_{k+2} + \cdots + b_n v_n) = 0$$

$$\Rightarrow \underbrace{b_{k+1} v_{k+1} + \cdots + b_n v_n}_{= c_1 v_1 + c_2 v_2 + \cdots + c_k v_k} \in N(T)$$

$\vdots$

$$b_{k+1} = b_{k+2} = \cdots = b_n = 0$$