MATH 430
Advanced Linear Algebra

Session 11

One-to-one & Onto transformations

$$T : V \longrightarrow W$$

T is one-to-one if for any $x_1, x_2 \in V$

$$T(x_1) = T(x_2) \Longrightarrow x_1 = x_2 .$$

Eg. $f : \mathbb{R} \longrightarrow \mathbb{R}$ (1) $f(x) = 2x + 3$ is one-to-one

$$f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$$

(2) $f(x) = x^2$ is NOT one-to-one.

$$x_1^2 = x_2^2 \not\Longrightarrow x_1 = x_2$$

$x_1 = 2 \rightarrow 4$

$x_2 = -2 \nearrow$

$T: V \longrightarrow W$ is <u>onto W</u> if for every $w \in W$, $\exists\, v \in V$ s.t. $T(v) = w$.

$f: \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2$ is NOT ONTO $\mathbb{R}$. Take $y = -4$, then $\nexists$ any $x \in \mathbb{R}$ s.t. $x^2 = -4$.

Facts that will be used :

- $\{\vec{0}\}$ is a subspace of V.
- $\{\vec{0}\} := \text{span}\{\phi\}$
- $\dim\{\vec{0}\} = 0$

Example $T : \overset{V}{P_2(\mathbb{R})} \longrightarrow \overset{W}{P_3(\mathbb{R})}$ codomain

$$T(f(x)) = f'(x) + \int_0^x 2f(t)\,dt$$

Basis of $P_2(\mathbb{R}) = \{1, x, x^2\}$; $\dim(P_2) = 3$. $\dim(V)$

$$R(T) = \text{span}\{T(1), T(x), T(x^2)\}$$

$$T(1) \overset{f\equiv 1}{=} \frac{d}{dx}(1) + 2\int_0^x 1 \cdot dt = 2x$$

$$T(x) \overset{f=x}{=} \frac{d}{dx}(x) + 2\int_0^x t\,dt = 1 + x^2$$

$$T(x^2) \overset{f=x^2}{=} \frac{d}{dx}(x^2) + 2\int_0^x t^2\,dt = 2x + \frac{2}{3}x^3$$

$$R(T) = \text{span}\underbrace{\left\{2x,\ 1+x^2,\ 2x+\frac{2}{3}x^3\right\}}_{\text{a l.i. set}} \qquad \text{rank}(T) = 3$$

S is a set of non zero polynomials such that no two have the same degree.

S is linearly independent.

Try to show this as hw.

$$\dim(N(T)) \underset{\substack{\uparrow \\ \text{Dimension Theorem}}}{=} \dim(V) - \operatorname{rank}(T) = 3 - 3 = 0$$

$N(T) = \{\vec{0}\}$ ⟶ zero polynomial

$\Rightarrow$ only the zero polynomial gets mapped to the zero polynomial $\Rightarrow$ T is one-one (due to a Theorem we will discuss)

$\dim(R(T)) = 3$, $\dim(P_3) = 4$

$R(T) \subseteq P_3$; $R(T) \neq P_3$ because $3 \neq 4$

$\Rightarrow$ T is <u>NOT onto</u>

Theorem: $T : V \to W$ is a linear transformation.

Then T is one-to-one $\overset{\text{if and only if}}{\Longleftrightarrow}$ $N(T) = \{\vec{0}\}$

Proof: $(\Rightarrow)$ Let T be one-to-one.

Want to show that $N(T) = \{\vec{0}_V\}$.

Since T is linear, $T(\vec{0}_V) = \vec{0}_W$

Let $x \in N(T)$. Then $T(x) = \vec{0}_W$

Therefore $T(x) = T(\vec{0}_V)$.

Since T is one-to-one, $x = \vec{0}_V$.

$\Rightarrow N(T) = \{\vec{0}\}$.

($\Leftarrow$:) Let $N(T) = \{\vec{0}_V\}$. Want to show that T is one-to-one.

Let $T(x_1) = T(x_2)$

$\Rightarrow \quad T(x_1) - T(x_2) = \vec{0}_W$

$\Rightarrow \quad T(x_1 - x_2) = \vec{0}_W$

T is linear

$\Rightarrow \quad x_1 - x_2 \in N(T)$.

$\Rightarrow \quad x_1 - x_2 = \vec{0}_V$

$\Rightarrow \quad x_1 = x_2 \quad \Rightarrow T$ is one-to-one.

Ques : $T : V \longrightarrow W$

Basis of $V = \{ v_1, \dots, v_m \}$; $\dim(V) = m$.

$R(T) = \text{span}\{ T(v_1), \dots, T(v_m) \}$

Let T be one-one.

Is $\{ T(v_1), \dots, T(v_m) \}$ a basis of $R(T)$?

Answer : Yes. T is 1-1 $\Rightarrow$ nullity $= \boxed{0}$.

By Dim. Thm. $\underbrace{\text{rank}(T)}_{\dim(R(T))} = m - 0 = m$

The spanning set of $R(T)$ i.e. $\{ T(v_1) \dots T(v_m) \}$ must be a basis as it has m elements.