University of Idaho

MATH 430
Advanced Linear Algebra

Session 12

One-to-one and onto transformations (Continued):

Last time: Theorem $\left[\begin{array}{l} T : V \longrightarrow W \text{ is one-to-one } \Longleftrightarrow \\ N(T) = \{\vec{0}\} \end{array} \right.$

In a special case, <u>one-to-one</u> and <u>onto</u> are equivalent.

<u>Theorem</u>: Let $\boxed{\dim(V) = \dim(W) < \infty}$, $T : V \to W$ be linear. Then the following are <u>equivalent</u>:

(a) T is one-to-one

(b) T is onto

(c) $\underbrace{\text{rank}(T)}_{\dim R(T)} = \dim(V)$

(a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c)

(a) T is one-to-one $\iff N(T) = \{\vec{0}_V\}$

$\iff \text{nullity}(T) = 0$

$\iff$ (c) $\text{rank}(T) = \dim(V)$ → Dimension Thm

$\iff \text{rank}(T) = \dim(W)$ → $\dim(V) = \dim(W)$

$\iff \dim(R(T)) = \dim(W)$

$\iff$ (b) $R(T) = W \iff T$ is onto.

subspace

$R(T) \subseteq W$

Remark : If V is not finite dimensional then one-to-one and onto need not be equivalent. (HW 4, #4)

Theorem:

Let $\{v_1, v_2, \dots, v_n\}$ be a **basis** of V.
Given $w_1, w_2, \dots, w_n$ in W, there exists a **unique** linear transformation $T : V \longrightarrow W$ s.t.
$T(v_i) = w_i$ for $i = 1, 2, \dots, n$.
[Some w_is may be repeated]

Proof: Let $x \in V$. Then $\exists$ unique scalars $a_1, a_2, \dots, a_n$ s.t. $x = \sum_{i=1}^{n} a_i v_i$.

Define $T : V \longrightarrow W$ by $T(x) = \sum_{i=1}^{n} a_i w_i$

Claim: This is the required T.

(a) T is linear. Show $T(cx+y) = c\,T(x) + T(y)$.

(b) Let $x = v_i$. Then $a_i = 1$, $a_j = 0$ for $j \neq i$.

$\Rightarrow$ $T(v_i) = a_i w_i = 1 \cdot w_i = w_i$

(c) T is unique: Let $S : V \longrightarrow W$ be another linear transformation s.t. $S(v_i) = w_i$, $i = 1, \dots, n$.

Then for $x = \sum_{i=1}^{n} a_i v_i \in V$

$$S(x) = S\left(\sum_{i=1}^{n} a_i v_i\right) \overset{S \text{ is linear}}{=} \sum_{i=1}^{n} a_i S(v_i) = \sum_{i=1}^{n} a_i w_i$$

$$= T(x).$$

Since x is arbitrary, $S = T$ $\square$

Earlier : $T : V \to W$, T is given explicitly.

We know a basis of V : $\{v_1, v_2, \dots, v_n\}$

Then $R(T) = \text{span}\{T(v_1), T(v_2), \dots, T(v_n)\}$

Any $x \in V$ can be uniquely written as

$$x = \sum_{i=1}^{n} a_i v_i$$

Then

$$T(x) = T\left(\sum a_i v_i\right) = \sum_{i=1}^{n} a_i \underbrace{T(v_i)}_{\text{known}}$$

Now: T is NOT given; we know a basis of V: $\{v_1, \dots, v_n\}$, and we are given n points in W: $w_1, \dots, w_n$.

Then $\exists$ a unique T s.t. $T(v_i) = w_i$

and any $\underline{x} = \sum_{i=1}^{n} a_i v_i$ is mapped to

$$\boxed{T(x) = \sum_{i=1}^{n} a_i w_i}$$

$$T(x) = T\left(\sum a_i v_i\right) = \sum a_i \underbrace{T(v_i)}_{w_i}$$

Example: $V = \mathbb{R}^2$ Take $\{a = (1,2),\ b = (3,4)\}$ in $\mathbb{R}^2$.
This forms a basis of $\mathbb{R}^2$. Given $(3,2,1)$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$
$(6,5,4)$ in $\mathbb{R}^3$. Then there is a unique T
s.t. $T(1,2) = (3,2,1)$ and $T(3,4) = (6,5,4)$.
This defines T. We can find $T(x)$ for any $x \in \mathbb{R}^2$

$T(1,0) = ?$

$c_1 = -2,\quad c_2 = 1$

(i) $(1,0) = c_1(1,2) + c_2(3,4)$
$= -2(1,2) + 1\cdot(3,4)$

(ii) $T(1,0) = -2\begin{pmatrix}3\\2\\1\end{pmatrix} + \begin{pmatrix}6\\5\\4\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}.$

Is T one-to-one? yes $\quad T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$R(T) = \text{span}\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} \right\} \Rightarrow \dim(R(T)) = 2$

$\Rightarrow \text{rank}(T) = 2 \overset{\text{dimension Thm}}{\Longrightarrow} \text{nullity} = 0$

onto? No $\quad R(T) \neq \mathbb{R}^3$