MATH 430
Advanced Linear Algebra

Session 13

Matrix of a linear transformation (2.2)

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is given by $\boxed{T(x) = Ax}$ where $\boxed{A \in M_{m \times n}(\mathbb{R})}$, Matrix transformation

then T is linear.

$T(x+y) = T(x) + T(y)$
$A(x+y) = Ax + Ay$
$T(cx) = c\,T(x)$
$A.(cx) = c\,Ax$

Every linear transformation in a finite dimensional setting has a matrix representation.

$$T : V \longrightarrow W$$
$$x \longmapsto y$$

Let $\dim(V) = n$ and $\dim(W) = m$.

$\alpha = \underbrace{\{v_1, v_2, \dots, v_n\}}_{\text{order matters}}$ is a basis of V.

order matters ; α is an ordered basis

$\beta = \{w_1, w_2, \dots, w_m\}$ is an ordered basis of W.

$x \in V$; $x = \sum_{i=1}^{n} x_i v_i$. Let $\underline{T(x) = y} \in W$

$y = \sum_{i=1}^{m} y_i w_i$

Think of x as $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

Think of y as $(y_1, y_2, \dots, y_m) \in \mathbb{R}^m$.

Goal : Find a matrix A such that

$$\underset{m \times n}{A} \begin{array}{c} x \\ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ \downarrow \\ \text{w.r.t. } \alpha \end{array} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \hookrightarrow \text{w.r.t. } \beta$$

A is the matrix of T : $[T]_{\alpha}^{\beta} = A$

Example (1) $\mathcal{U} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $(x_1, x_2) \in \mathbb{R}^2$

$$\boxed{\mathcal{U}(x_1, x_2) = (x_1, 0, 0)}$$

desired matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$$

Basis of $\mathbb{R}^2$: $\{ \underset{e_1}{(1,0)}, \underset{e_2}{(0,1)} \}$

Basis of $\mathbb{R}^3$: $\left\{ \underset{s_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{s_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{s_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$

$\mathcal{U}(e_1) = (1,0,0) = 1 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$

$\mathcal{U}(e_2) = (0,0,0) = 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$

$$[\mathcal{U}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example (2) $T: P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$T(f(x)) = f'(x)$ T: differential operator

Basis of P_3 : $\{1, x, x^2, x^3\}$: fixed order

Basis of P_2 : $\{1, x, x^2\}$: ordered basis

$$\left.\begin{aligned}
T(1) &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\
T(x) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\
T(x^2) &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\
T(x^3) &= 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2
\end{aligned}\right\}$$

write the images in terms of the basis vectors of $P_2(\mathbb{R})$.

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \sim (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$

$T(f(x)) = a_1 + 2a_2 x + 3a_3 x^2 \sim (a_1, 2a_2, 3a_3) \in \mathbb{R}^3$

$$[T]\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}$$

Example 3

$\mathcal{U} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\mathcal{U}(x_1, x_2) = (x_1, 0, 0)$ $\quad [\mathcal{U}] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad T(x_1, x_2) = (-x_2, 0)$

$[T]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ 0 \end{pmatrix}$

$\mathcal{U} \circ T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad \mathcal{U}(T(x_1, x_2)) = \mathcal{U}((-x_2, 0))$

$= (-x_2, 0, 0)$

$[\mathcal{U} \circ T] = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{\text{check}}{=} [\mathcal{U}][T]$

Find by taking the bases of $\mathbb{R}^2$, $\mathbb{R}^3$

Composition of transformations & matrix multiplication

$T : \overset{n}{V} \longrightarrow \overset{m}{W}$, $U : \overset{m}{W} \longrightarrow \overset{r}{Z}$ are linear

$U \cdot T : V \longrightarrow Z$ is also linear.

$$V \xrightarrow{T} W \xrightarrow{U} Z$$

$$v \longmapsto \underset{= T(v)}{w} \longmapsto z = U(w) = U(T(v)).$$

Theorem : $[U \cdot T]_{r \times n} = [U]_{r \times m} [T]_{m \times n}$