MATH 430
Advanced Linear Algebra

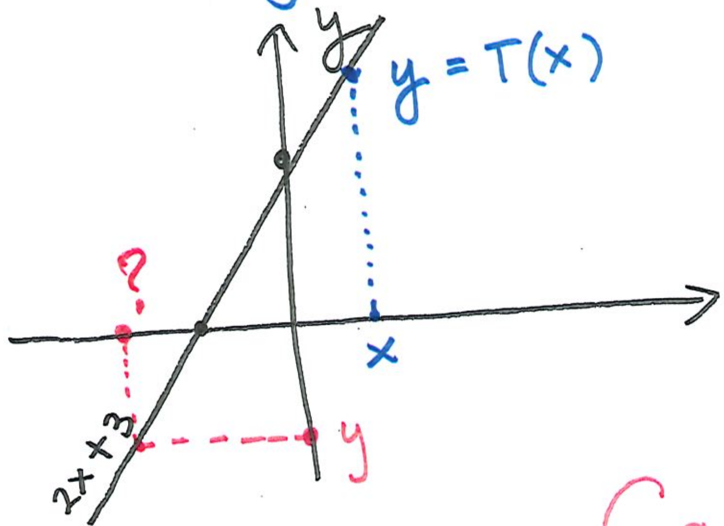
Session 14

Invertibility and Isomorphism

$T : \mathbb{R} \longrightarrow \mathbb{R}$

$y = T(x) = \boxed{2x+3}$

Given y, find x
s.t. $T(x) = y$.
If such an x exists
then
$x = T^{-1}(y)$



$f(x) = x^2$
$f : \mathbb{R} \to \mathbb{R}$
Then f^{-1}
does not
exist

$x = \dfrac{y-3}{2}$

$$\boxed{T^{-1}(y) = \frac{y-3}{2}}$$

$T^{-1} : \mathbb{R} \to \mathbb{R}$

T is invertible &
T^{-1} exists

Caution -

$$T^{-1}(y) \neq \frac{1}{T(y)}$$

T	T^{-1}
e^x	$\ln(y)$
$\ln x$	e^y
$2x+3$	$\dfrac{y-3}{2}$

$T : V \longrightarrow W$ is linear

The inverse of T, denoted by T^{-1}, is a mapping

$$T^{-1} : W \longrightarrow V$$

such that $T^{-1}(\vec{w}) = \vec{v} \iff T(\vec{v}) = \vec{w}$

$T \circ T^{-1}(w) = T(v) = w$ $\qquad$ $T^{-1}(T(v)) = \underbrace{T^{-1}(w)}_{=v}$

$I_V : V \to V$

$I(\vec{v}) = \vec{v}$

Identity transformation

T^{-1} is such that $T^{-1} \circ T = I_V$

and $T \circ T^{-1} = I_W$

$T : V \to W$

$T^{-1} : W \to V$

If T^{-1} exists, then T is said to be <u>invertible</u>.

- T is invertible $\Longleftrightarrow$ T is one-to-one and onto
- If T is linear and invertible then T^{-1} is also linear.
  (see proof in the book)
- If T^{-1} exists, then it is unique.
- <u>Theorem</u>: If T^{-1} exists, then $[T^{-1}] = [T]^{-1}$

  matrix inverse

Example: $T : \underset{V}{P_1(\mathbb{R})} \longrightarrow \overset{W}{\mathbb{R}^2}$

$T(\underbrace{a+bx}_{\in P_1}) = (\overbrace{a, a+b}^{\in \mathbb{R}^2})$

$T^{-1}(\underbrace{c, d}_{\in \mathbb{R}^2}) = c + (d-c)x$

$\dim(\mathbb{R}^2) = 2$

$\dim(P_1) = 2$

$P_1 \sim \mathbb{R}^2$

T is an isomorphism

$1 = 1 + 0 \cdot x$

$a+bx \longmapsto (a, b)$

another isomorphism

Take $\{1, x\}$ as the basis of P_1, $\{e_1, e_2\}$ as the basis of $\mathbb{R}^2$

$T(1) = (1, 1) = 1 \cdot e_1 + 1 \cdot e_2$

$T(x) = (0, 1) = 0 \cdot e_1 + 1 e_2$.

$[T] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$[T^{-1}] = [T]^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = A$

Find the matrix of T^{-1} using the standard process, and check that it is the same as A.

Isomorphism

$T : V \to W$ is linear.

If T is invertible (i.e. one-one & onto) then T is said to be an <u>isomorphism</u> from V to W. We also say that V and W are isomorphic. Notation: $V \sim W$

Theorem: Let V & W be finite dimensional. V is isomorphic to $W \iff \dim(V) = \dim(W)$.

$V \sim W \iff \exists\, T : V \to W$ that is one-one & onto

$\iff \dim(N(T)) = 0$ and $\dim(R(T)) = \dim(W)$

By dimension Theorem $\dim(R(T)) = \dim(V)$

Vector space of linear transformations

Addition : $T, U : V \longrightarrow W$

$T + U : V \longrightarrow W$ is defined as

$$\underbrace{(T+U)}_{\text{sum}}(x) = T(x) + U(x) \quad \forall x \in V.$$

Scalar multiplication : $cT : V \longrightarrow W$ is given

$c \in \mathbb{F}$ by $\underbrace{(cT)}_{\text{scalar multiple of } T.}(x) = c\,T(x) \quad \forall x \in V$

see the book for proof

Under the above operations the collection of linear transformations from $V \longrightarrow W$ is a vector space, denoted by $\mathcal{L}(V, W)$. If $V = W$, then we write $\mathcal{L}(V)$.