MATH 430
Advanced Linear Algebra

Session 15

Change of basis

Example : $V = \mathbb{R}^2$ $\quad x = (5,8) \in \mathbb{R}^2$.

The standard basis is $\left\{ \underset{e_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{e_2}{\begin{pmatrix}0\\1\end{pmatrix}} \right\} =: \alpha$

$(5,8) = 5\begin{pmatrix}1\\0\end{pmatrix} +$
$8\begin{pmatrix}0\\1\end{pmatrix}$

$(5,8) = ⑤e_1 + ⑧e_2$

Then $[x]_\alpha = (5,8)$.

Take another basis : $\beta = \left\{ \overset{b_1}{\begin{pmatrix}1\\2\end{pmatrix}}, \overset{b_2}{\begin{pmatrix}4\\7\end{pmatrix}} \right\}$.

Find $[x]_\beta$.

We need to find c_1, c_2 such that

$$c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}4\\7\end{pmatrix} = 5\begin{pmatrix}1\\0\end{pmatrix} + 8\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{bmatrix}1 & 4\\2 & 7\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}5\\8\end{bmatrix}$$

$$\boxed{[x]_\beta = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1 & 4\\2 & 7\end{bmatrix}^{-1} \underbrace{\begin{bmatrix}5\\8\end{bmatrix}}_{[x]_\alpha}} = \begin{bmatrix}-7 & 4\\2 & -1\end{bmatrix}\begin{bmatrix}5\\8\end{bmatrix} = \begin{bmatrix}-3\\2\end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}^{-1} = -1 \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

$$\begin{array}{cc} b_1 & b_2 \\ \left[\begin{matrix} 1 \\ 2 \end{matrix}\right. & \left.\begin{matrix} 4 \\ 7 \end{matrix}\right] \end{array} = Q$$

$$[x]_\beta = Q^{-1} [x]_\alpha \iff Q [x]_\beta = [x]_\alpha$$

We can toggle between α & β.

Q : change of coordinate matrix from β to α

Q^{-1} : " " " from α to β

Generalize the above to $\mathbb{R}^n$: $V = \mathbb{R}^n$, $x \in \mathbb{R}^n$

$[x]$: coordinates of x w.r.t. the std. basis of $\mathbb{R}^n$.

Let $\beta = \{b_1, b_2, \ldots, b_n\}$ be a new basis of $\mathbb{R}^n$.

$$Q = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{bmatrix} \quad ; \quad [x]_\beta = Q^{-1}[x] \quad \text{and}$$

$$[x] = Q\,[x]_\beta \, .$$

Example : $V = \mathbb{R}^2$, $x \in V$. Let $\alpha = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$ and $\beta = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \end{pmatrix} \right\}$ be two bases of $\mathbb{R}^2$.

Given $[x]_\alpha = (-2, 6)$. Find $[x]_\beta$.

Need to find $\boxed{c_1 , c_2}$ s.t.

$$-2\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 6\begin{pmatrix} 1 \\ 4 \end{pmatrix} = c_1\begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$\underset{A_\alpha}{\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}} \underset{[x]_\alpha}{\begin{bmatrix} -2 \\ 6 \end{bmatrix}} = \underset{A_\beta}{\begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix}} \underset{[x]_\beta}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\underbrace{[x]_\beta}_{\text{unknown}} = \boxed{A_\beta^{-1} A_\alpha} \underbrace{[x]_\alpha}_{\text{known}}$$

$$\Updownarrow$$

$$[x]_\alpha = \boxed{A_\alpha^{-1} A_\beta} [x]_\beta$$

change of coordinate matrix

Finding the matrix of a linear transformation w.r.t. a new basis : What is $[T]_\beta$ in terms of $[T]_\alpha$

$T : V \longrightarrow V$ Let α, β be two bases of V.

Suppose $[T]_\alpha$ is known.

Find $[T]_\beta$.

Example : $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ is given by $T(f(x)) = f'(x)$.

$\alpha = \{x^2, x, 1\}$ is a basis of $P_2(\mathbb{R})$.

$$T(x^2) = 2x = 0\cdot x^2 + 2\cdot x + 0\cdot 1$$
$$T(x) = 1 = 0\cdot x^2 + 0\cdot x + 1.1$$
$$T(1) = 0 = 0\cdot x^2 + 0\cdot x + 0\cdot 1$$

$$[T]_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\beta = \{2x^2-x, 3x^2+1, x^2\}$ is another basis of $P_2(\mathbb{R})$. Find $[T]_\beta$.

Consider a general vector in $P_2(\mathbb{R})$: $\underbrace{\overset{f(x)}{ax^2+bx+c}}_{\text{w.r.t. } \alpha}$

~~$ax^2+bx+c = r_1(2x^2-x), 3x^2+1, x^2$~~

$$ax^2+bx+c = r_1(2x^2-x) + r_2(3x^2+1) + r_3x^2$$

$$\left.\begin{aligned} a &= 2r_1 + 3r_2 + r_3 \\ b &= -r_1 \\ c &= r_2 \end{aligned}\right] \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \overbrace{\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}}$$

$f \sim (a,b,c) \sim (r_1, r_2, r_3)$

$$[f]_\alpha = Q\,[f]_\beta$$

$$\underbrace{[T]_\alpha [f]_\alpha}_{\text{Image of } f \text{ in } \alpha} = [T]_\alpha \, Q \, [f]_\beta$$

Image of f in β is

$$\underbrace{Q^{-1} [T]_\alpha Q}_{[T]_\beta} [f]_\beta = [T]_\beta [f]_\beta$$

$$\circledast \quad [T]_\beta = Q^{-1} [T]_\alpha Q .$$

Find $[T]_\beta$ the <u>usual way</u> and show that it is the same as $\circledast$