University of Idaho

MATH 430
Advanced Linear Algebra

Session 17

Systems of linear equations (Ch: 3) ①

$$A\,x = b$$

Example 1.

$$x_1 + 2x_2 + x_3 = 1$$
$$x_1 + x_2 + x_3 = 1$$
$$2x_1 + 3x_2 + 3x_3 = 1$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b}$$

$R2 \longrightarrow R2 - R1$
$R3 \longrightarrow R3 - 2R1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$R3 \longrightarrow R3 - R2$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\hat{b}}$$

U - upper triangular
$Ax = b$ and $Ux = \hat{b}$ have the same solution.

$x_3 = -1$, $x_2 = 0$

R1 : $x_1 + 2x_2 + x_3 = 1 \Rightarrow x_1 = 2$

$\boxed{\text{UNIQUE SOLUTION}}$

Ex. 2. Change the above slightly

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{b}$$

Use the same row operations to get

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\hat{b}}$$

R3 $\Rightarrow 0 = -1$ which is NOT possible

There is <u>no solution</u>; the system is <u>inconsistent</u>.

Ex. 3 Change Example 1 slightly and consider

$$\underset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}$$

The same row operations give

$$\underset{U}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\hat{b}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

R1: $x_1 + 2x_2 + x_3 = 1$

R2: $-x_2 = 0 \Rightarrow x_2 = 0$

R3: $0 = 0$ ✓

$x_1 + x_3 = 1$

Pick x_3 as you like. Let $x_3 = \alpha$. Then $x_1 = 1-\alpha$

Solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ 0 \\ \alpha \end{bmatrix}$; $\alpha \in \mathbb{R}$.

There are infinitely many solutions, one for each α.

In general: given $b \in \mathbb{R}^m$, $A \in M_{m \times n}$.

Find $x \in \mathbb{R}^n$ s.t. $\boxed{Ax = b}$

$$\boxed{Ax = b} \xrightarrow{\text{row reductions}} \boxed{Ux = \hat{b}}$$

U is upper triangular or almost upper triangular: echelon form

$Ax = b$ and $Ux = \hat{b}$ have the same solution.

$$U = \begin{bmatrix} \boxed{p_1} & \times & \times & \cdots & \times \\ 0 & 0 & \boxed{p_2} & \times \cdots & \times \\ 0 & 0 & 0 & \boxed{p_3} \cdots & \times \\ & & & \ddots & \end{bmatrix}$$

m (rows), n (columns)

pivots

$p_1 \neq 0$
$p_2 \neq 0$
$p_3 \neq 0$
$\vdots$

$p_1, p_2, p_3 \cdots$ are the leading non-zero entry in each row.

Rank of a matrix A = maximum number of linearly independent columns

\# pivots = max. number of l.i. columns of U
= $\boxed{\text{rank of } U = \text{rank}(A)}$

Elementary row operations preserve the rank.

Suppose U is $n \times n$ (square) upper triangular with nonzero diagonal entries, i.e., U has n pivots then rank of $U = n$ and U has n l.i. columns. Review HW2, #4. (show that the cols. of an upper triangular matrix with nonzero diagonal entries are l.i.)

$$[A \mid b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$ Augmented matrix

$Ax = b$

$$[A \mid b] \xrightarrow{\text{row operations}} [U \mid \hat{b}]$$

When does $Ax = b$ have a solution?

Rewrite $Ax = b$ as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

col 1 of A — col 2 of A — col n of A — b

Solution to $Ax = b$ exists if and only if ($\Longleftrightarrow$)
$b \in \text{span}\{\text{cols. of } A\}$

$$\Longleftrightarrow \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \right\} \text{ is}$$

linearly dependent

$$\Longleftrightarrow \quad \text{rank}(A) = \text{rank}(A \mid b)$$