

MATH 430
Advanced Linear Algebra

Session 17

Example 1.

$$x_1 + 2x_2 + x_3 = 1$$

$$x_1 + x_2 + x_3 = 1$$

$$2x_1 + 3x_2 + 3x_3 = 1$$

$$R2 \rightarrow R2 - R1$$

$$R3 \rightarrow R3 - 2R1$$

$$R3 \rightarrow R3 - R2$$

U - upper triangular
 $Ax = b$ and $Ux = \hat{b}$
 have the same solution.

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad b$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \hat{b}$$

$$x_3 = -1, x_2 = 0$$

$$R1: x_1 + 2x_2 + x_3 = 1 \Rightarrow x_1 = 2$$

UNIQUE SOLUTION

Ex. 2. Change the above slightly

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Use the same row operations to get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$R_3 \Rightarrow 0 = -1$
which is NOT possible

There is no solution; the system is inconsistent.

Ex. 3

Change

$$A \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad b$$

Example 1 slightly and consider

The same row operations give

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{b}$$

$R1: x_1 + 2x_2 + x_3 = 1$
 $R2: -x_2 = 0 \Rightarrow x_2 = 0$
 $R3: 0 = 0 \checkmark$
 $x_1 + x_3 = 1$

Pick x_3 as you like. Let $x_3 = \alpha$. Then $x_1 = 1 - \alpha$

Solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - \alpha \\ 0 \\ \alpha \end{bmatrix}$; $\alpha \in \mathbb{R}$.

There are infinitely many solutions, one for each α .

In general: given $b \in \mathbb{R}^m$, $A \in M_{m \times n}$.

$$Ax = b$$

Find $x \in \mathbb{R}^n$ s.t.

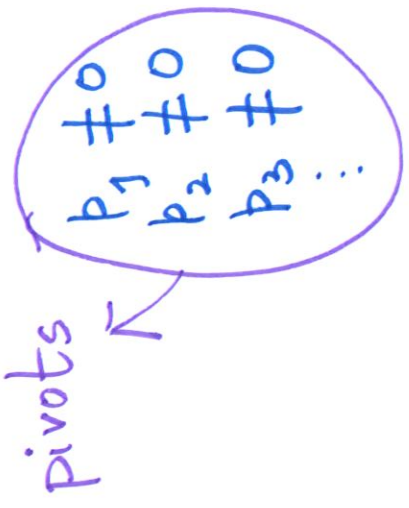
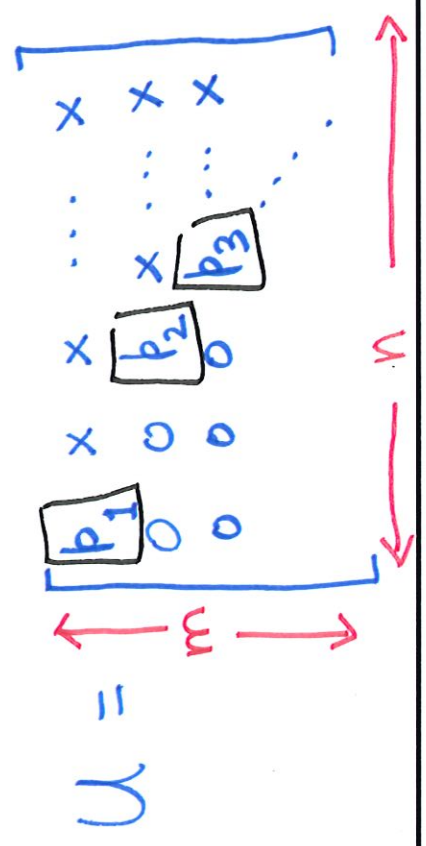
row reductions

$$Ux = \hat{b}$$

U is upper triangular or almost upper

triangular: echelon form

$Ax = b$ and $Ux = \hat{b}$ have the same solution.



p_1, p_2, p_3, \dots are the leading non-zero entry in each row.

Rank of a matrix A = maximum number of linearly independent columns

$$\begin{aligned} \# \text{ pivots} &= \text{max. number of l.i. columns of } U \\ &= \boxed{\text{rank of } U = \text{rank}(A)} \end{aligned}$$

Elementary row operations preserve the rank.

Suppose U is $n \times n$ (square) upper triangular with nonzero diagonal entries, i.e., U has n pivots then rank of $U = n$ and U has n l.i. columns.

Columns. Review HW2, #4. (show that the cols. of an upper triangular matrix with nonzero diagonal entries are l.i.)

Augmented matrix

$$Ax = b$$

$$[A|b] = \begin{bmatrix} a_{11} & \dots & a_{1n} & | & b_1 \\ a_{21} & \dots & a_{2n} & | & b_2 \\ \vdots & \dots & \vdots & | & \vdots \\ a_{m1} & \dots & a_{mn} & | & b_m \end{bmatrix}$$

row operations

$$[U|\hat{b}]$$

When does $Ax = b$ have a solution? \downarrow

Rewrite $Ax = b$ as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

\uparrow Col 1 of A
 \uparrow Col 2 of A
 \uparrow Col n of A
 \uparrow b

\Leftrightarrow Solution to $Ax = b$ exists if and only if
 $b \in \text{span} \{ \text{cols. of } A \}$

$$\Leftrightarrow \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ is}$$

linearly dependent

$$\Leftrightarrow \text{rank}(A) = \text{rank}(A|b)$$