MATH 430
Advanced Linear Algebra

Session 18

Systems of linear equations

$Ax = b \qquad A \in M_{m \times n}, \quad b \in \mathbb{R}^m.$

Homogeneous if $b = \vec{0}$.

Non-homogeneous if $b \neq \vec{0}$.

Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be defined by $\varphi(x) = Ax$.

$$x \longmapsto Ax = b$$

Solutions of $\boxed{Ax = 0}$ form the null space of A, denoted by $N(A)$.

$x = \vec{0}$ is always a solution. There may be other solutions.

Range(A) or $R(A)$ = span of the columns of A.

Recall: $Ax = b$ has a solution $\Longleftrightarrow b \in \text{span}\{\text{cols. of } A\}$

max. # of l.i. columns of $A = \dim(R(A))$

$= \text{rank}(A)$

By the Dimension Theorem

$$\dim(\mathbb{R}^n) = \dim(N(A)) + \text{rank}(A)$$

$$\boxed{\dim(N(A)) = \dim(\mathbb{R}^n) - \text{rank}(A)}$$ Thm 3.8

<u>Thm 3.9</u> Let K be the solution set of $\underline{\underline{Ax=b}}$.

Let $x = k$ be a solution. Then

$$K = \{ \underline{\underline{s}} + k : \underline{\underline{s}} \in N(A) \}.$$

s solves $Ax = \vec{0}$

This means: any solution of $Ax = b$ can be expressed as the sum of a particular solution of $Ax = b$ and any solution of $\underline{\underline{Ax = \vec{0}}}$.

Example (from previous session):

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{b} \xrightarrow[\text{reduction}]{\text{row}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 1 - \alpha, \quad x_2 = 0, \quad x_3 = \alpha, \quad \alpha \in \mathbb{R}.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underbrace{\alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{s} = r + s$$

$r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (when $\alpha = 0$) solves the above system.

Check that $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ solves $Ax = \vec{0}$.

$$N(A) = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$r = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is called a particular solution.

$A \in M_{m \times n}$

Unique solution : $Ax = b$ has only one solution for a given b, i.e.

$$x \longmapsto \underset{\substack{\parallel \\ b}}{Ax} \quad \text{is one-to-one}$$

one-to-one $\Longleftrightarrow N(A) = \{\vec{0}\} \Longleftrightarrow Ax = \vec{0}$ has <u>only</u> the trivial solution $x = \vec{0}$.

In the above example, $N(A) \neq \{\vec{0}\}$

$\Rightarrow$ solution is not unique.

Let A be square $(n \times n)$: there are n equations in n unknowns.

$$\underset{\in \mathbb{R}^n}{x} \longmapsto \underset{\in \mathbb{R}^n}{Ax = b} \qquad \underbrace{A}_{n\times n}\,\underbrace{x}_{n\times 1} = \underbrace{b}_{n\times 1}$$

$$A : \overset{V}{\mathbb{R}^n} \longrightarrow \overset{W}{\mathbb{R}^n} \qquad V = W = \mathbb{R}^n \qquad \underset{\in \mathbb{R}^n}{x} \longmapsto \underset{\in \mathbb{R}^n}{b}$$

A is one-one $\Longleftrightarrow$ A is onto

$\Longleftrightarrow$ cols. of A span $\mathbb{R}^n$.

Then
A is invertible and the solution is

$$x = A^{-1} b.$$

(Review finding matrix inverses)

$$[A \mid I] \xrightarrow{\text{reduce}} [I \mid A^{-1}]$$

Summary : Solve $Ax = b$

(a) solution exists $\iff$ $\text{rank}(A) = \text{rank}[A \mid b]$

$\iff$ $b \in$ span {cols. of A}.

Column space of A

(b) $Ax = \vec{0}$ has only the trivial solution $x = \vec{0}$

$\iff$ $N(A) = \{\vec{0}\}$ $\iff$ $Ax = b$ has a unique solution.

(c) If A is in $M_{n \times n}$ and $N(A) = \{\vec{0}\}$, the unique solution is $x = A^{-1}b$.

(d) If $N(A) = \{\vec{0}\}$ and $A \in M_{n \times n}$ then $\text{rank}(A) = n$.