University of Idaho

MATH 430
Advanced Linear Algebra

Session 19

Determinants (Ch: 4)

Determinant of a matrix $A \in M_{n \times n}$ is denoted by $\det(A)$ or $|A|$. The determinant is only defined for square matrices.

Main use is in calculating eigenvalues of a matrix. Sometimes it is used in calculating the <u>inverse</u> of a matrix as well as solving linear systems by <u>Cramer's Rule</u>.

<u>Definition</u> If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}$, then

$$|A| = ad - bc$$

Determinant of an $n \times n$ matrix ; $n > 2$

$$|A| = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \end{array} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - eg) \rightarrow \circledast$$

expand about row 1.

Can expand about <u>any row</u> or <u>any column</u>.

Expand about col. 2

$$|A| = (-1)^{1+2} b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + (-1)^{2+2} e \begin{vmatrix} a & c \\ g & i \end{vmatrix} + (-1)^{3+2} h \begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

$$= -b(di - gf) + e(ai - gc) - h(af - dc)$$

which is same as $\circledast$.

Properties of determinants

- Let A^t be the transpose of A.

$$|A^t| = |A|$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

- If A has a row or column of zeros, then $|A| = 0$.

Determinant of special matrices:

(a) Upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$|A| \underset{\substack{\uparrow \\ \text{expand about} \\ \text{col. 1}}}{=} a_{11} \begin{vmatrix} a_{22} & \cdots & & a_{2n} \\ & a_{33} & \cdots & a_{3n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{vmatrix}$$

$|A| = a_{11} a_{22} a_{33} \cdots a_{nn}$ (product of the diagonal entries)

(b) Lower triangular

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad |A| = a_{11} a_{22} \cdots a_{nn}$$

For triangular matrices the determinant equals the product of the diagonal entries.

Diagonal matrices (which have non zero entries only on the diagonal; everything else is zero) are special cases of triangular matrices.

ⓒ Identity matrix : $I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$

$\det(I_n) = 1$

How do elementary row operations change the determinant?

① Adding a scalar multiple of a row to another row keeps the determinant <u>unchanged</u>.

$$\det \begin{bmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{bmatrix} = 1(30-8) - 3(18+8) - 3(-12-20)$$
$$= 40$$

$R2 \longrightarrow R2 + 3R1$ $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ -4 & 4 & -6 \end{bmatrix}$; $4R1 + R3$ $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & -18 \end{bmatrix}$

no change in det ; no change in det

$R3 \longrightarrow R3 - 4R2$ $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 0 & 10 \end{bmatrix} = U$ $\quad |U| = 40$

no change in det

$1 = 10/10$

Corollary : If A has two identical rows (or two identical columns) then $\det(A) = 0$.

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} \underset{r_1 \to r_1 - r_2}{=} \begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

2. Interchanging two rows $\Rightarrow$ determinant gets multiplied by $\underline{-1}$.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Even number of ~~inch~~ interchanges $\Rightarrow$ NO change in the det.

Odd number of interchanges $\Rightarrow$ det. is negated.

3. Multiply a <u>row</u> by a nonzero scalar <u>k</u>: the determinant is multiplied by k.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\det(B) = k \det(A)$$

- $\det(AB) = \det(A)\det(B)$.
- Let A be invertible. Suppose A is $n \times n$.

$$A^{-1}A = I_n$$

$$\det(A^{-1}A) = \det(I_n)$$

$$\det(A^{-1})\det(A) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

<u>Thm</u>: If A^{-1} exists, then $|A^{-1}| = \dfrac{1}{|A|}$.

<u>Thm</u>: A^{-1} exists $\Longleftrightarrow |A| \neq 0$