MATH 430
Advanced Linear Algebra

Session 2

Vector Space

The set of scalars called a field, denoted by $\mathbb{F}$.

see Appendix C

$\mathbb{F} = \mathbb{R}, \mathbb{C}$ $\quad \mathbb{Z}$ is not a field.

A <u>vector space</u> V over a field $\mathbb{F}$ is a set on which two operations called addition ($\oplus$) and scalar multiplication ($\odot$) are defined so that V is closed under addition and scalar multiplication and for $\vec{x}, \vec{y}, \vec{z} \in V$, $a, b \in \mathbb{F}$

VS1 : $\vec{x} \oplus \vec{y} = \vec{y} \oplus \vec{x}$ commutative

VS2 : $(\vec{x} \oplus \vec{y}) \oplus \vec{z} = \vec{x} \oplus (\vec{y} \oplus \vec{z})$ associative

VS3 : $\exists$ a unique element $\vec{0} \in V$ such that additive identity

$\vec{x} \oplus \vec{0} = \vec{x} \quad \forall \vec{x} \in V$

VS4 : For each $\vec{x} \in V$, $\exists$ a $\vec{y} \in V$ s.t. additive inverse

$\vec{x} \oplus \vec{y} = \vec{0}$

VS5 : For each $\vec{x} \in V$, $1 \odot \vec{x} = \vec{x}$

VS6: $(ab) \odot \vec{x} = a \odot (b \odot \vec{x})$

VS7: $(a+b) \odot \vec{x} = a \odot \vec{x} \oplus b \odot \vec{x}$

VS8: $a \odot (\vec{x} \oplus \vec{y}) = a \odot \vec{x} \oplus a \odot \vec{y}$

Elements of V will be referred to as <u>vectors</u>.

Examples of vector spaces

① $V = \mathbb{R}^n$ or $\mathbb{C}^n$ $(n \geq 1)$ $\quad \mathbb{F} = \mathbb{R}, \mathbb{C}$

Elements of V are n-tuples $(a_1, a_2, \dots, a_n)$

$a_i \in \mathbb{R}, \mathbb{C}$

Addition $\quad \vec{x} = (x_1, \dots, x_n) \quad \vec{y} = (y_1, \dots, y_n)$

$$\vec{x} \oplus \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

Scalar multiplication : $c \in \mathbb{F}$

$$c \odot \vec{x} = (cx_1, cx_2, \dots, cx_n)$$

Zero vector $= (0, 0, \dots, 0)$

rows $\quad$ colums

(2) Set of $m \times n$ matrices, denoted by $M_{m\times n}$ is a vector space.

$\oplus$ Addition - matrix addition

$\odot$ scalar multiplication : multiplying a matrix by a scalar

Zero vector : $m \times n$ matrix of all zeros

(3) Polynomials :

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$a_0, a_1, \ldots, a_n \in \mathbb{F}$, $a_n \neq 0 \Rightarrow$ degree $= n$

$P(\mathbb{F})$: the set of <u>all</u> polynomials

$P(\mathbb{F})$ is a vector space.

Addition : $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

$a_n \neq 0$ $g(x) = b_0 + b_1 x + \cdots + b_m x^m$

$b_m \neq 0$

$m \leq n$ $f \oplus g = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_m+b_m)x^m$

$c \in \mathbb{F}$ $+ a_{m+1}x^{m+1} + \cdots + a_n x^n$

$c \odot f = ca_0 + ca_1 x + \cdots + ca_n x^n$

Zero polynomial $= \vec{0} = 0$ (polynomial with all zero coeffs.)

Deg. of $\vec{0}$ (zero polynomial) $= -1$ (by definition)

Degree of $f \oplus g$

$f(x) = 3x^2 + x + 5$

$g(x) = -3x^2 + 2x$

$f \oplus g = 3x + 5$

degree of $f \oplus g$

$\leq \max(\deg f, \deg g)$

+ Example of a set that is NOT a vector space $\mathbb{F} = \mathbb{R}$

$V = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$ — plane

Consider $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in V.

Define

$$x \oplus y = (x_1 + y_1, x_2 - y_2)$$

$$c \odot \vec{x} = (cx_1, cx_2)$$

$$y \oplus x = (y_1 + x_1, y_2 - x_2)$$

$x_2 - y_2 \neq y_2 - x_2$ in general, meaning

$x \oplus y \neq y \oplus x \Rightarrow$ VS1 fails.

Theorem [Cancellation law for vector addition]

If $\vec{x}$, $\vec{y}$, $\vec{z}$ are vectors in V such that

$$\vec{x} \oplus \vec{z} = \vec{y} \oplus \vec{z}$$

then $\vec{x} = \vec{y}$

Proof: $\vec{x} = \vec{x} \oplus \vec{0}$

There exists $\vec{v}$ s.t. $\vec{z} \oplus \vec{v} = \vec{0}$

$$\vec{x} \overset{VS3}{=} \vec{x} \oplus \vec{0} = \vec{x} \oplus (\vec{z} \oplus \vec{v})$$

$$\overset{VS2}{=} (\vec{x} \oplus \vec{z}) \oplus \vec{v} = (\vec{y} \oplus \vec{z}) \oplus \vec{v}$$

$$= \vec{y} \oplus (\vec{z} \oplus \vec{v}) = \vec{y} \oplus \vec{0}$$

$$= \vec{y} \qquad \square$$