University of Idaho

MATH 430
Advanced Linear Algebra

Session 20

Invertibility and determinant

A is n x n

$$[A \mid I] \longrightarrow [I \mid A^{-1}]$$

A^{-1} exists.

$$|A^{-1}| = \frac{1}{|A|}$$

(1) A^{-1} exists $\iff$ (2) A can be row reduced to I

$|I| = 1 \neq 0$

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$$

$\iff$ (3) rank(A) $= n$ $\iff$ (4) $|A| \neq 0$

$Ax = 0$ has only the trivial solution and A is $\underline{n}$ l.i. columns

(1) A^{-1} ~~ex~~ does not exist

$\iff$ (2) A can be reduced to U where U has a row of zeros.

$|U| = 0$

$\iff$ (3) rank(A) $< n$ $\iff$ (4) $|A| = 0$

$Ax = 0$ has one or more free variables and A has fewer than n l.i. columns

Eigenvalues & eigenvectors.

Matrix eigenvalue problem : Find scalars λ and nonzero vectors $\vec{x}$ in $\mathbb{R}^n$ such that

$$A\vec{x} = \lambda \vec{x}, \text{ for a given } A \in M_{n \times n}$$

λ : eigenvalue of A

$\vec{x}$: eigenvector of A corresponding to λ.

Example : $A = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$

Trace $(A) = 3+3 = 6$

Sum of e-values $= -3+9 = 6$

Find scalars λ and vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that

$$\begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\lambda I_{2\times 2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 3-\lambda & 6 \\ 6 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Nonzero solution $\Rightarrow \text{rank}\left(\begin{bmatrix} 3-\lambda & 6 \\ 6 & 3-\lambda \end{bmatrix} \right) < 2$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 6 \\ 6 & 3-\lambda \end{vmatrix} = 0 .$$

$$\Rightarrow (3-\lambda)^2 - 36 = 0 \Rightarrow \overbrace{\lambda^2 - 6\lambda - 27}^{\text{polynomial in } \lambda \text{ of degree } 2} = 0$$

$$\Rightarrow \lambda = -3, 9$$

$\lambda = -3$

$$\begin{bmatrix} 3-(-3) & 6 \\ 6 & 3-(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2 \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for $\lambda = -3$

Normalize v_1 : $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

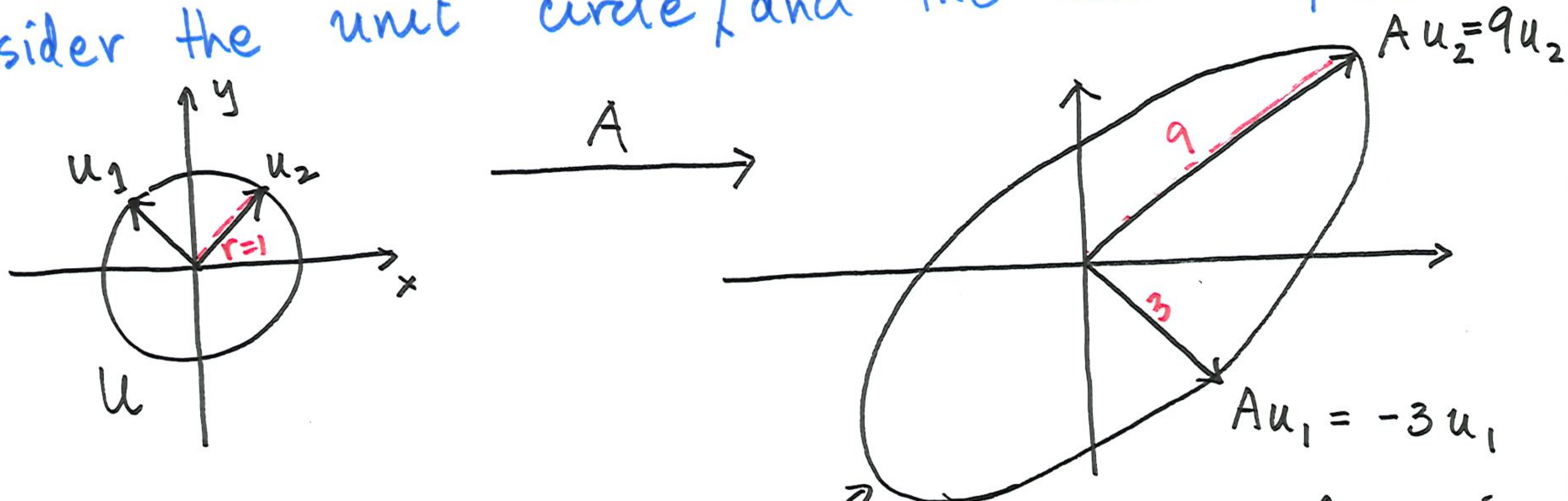
$\lambda = 9$ $\quad \begin{bmatrix} 3-9 & 6 \\ 6 & 3-9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 9$.

normalize v_2 :

$u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Consider the unit circle $\mathcal{U}$ and the action of A on $\mathcal{U}$.



ellipse with minor and major axis of lengths 3 & 9 respectively

For any matrix $A \in M_{n\times n}$:

Trace (A) = Sum of its eigenvalues.

Matrix e-value problem : $A \in M_{n \times n}$

Solve $\boxed{Ax = \lambda x}$

λ : unknown scalar

x : unknown vector.

Find scalar λ and vector $x \in \mathbb{R}^n$.

λ is an evalue $\iff$ $(A - \lambda I)X = 0$ has a nonzero solution $x \neq 0$.

$\iff x \in N(A - \lambda I)$, $x \neq 0$

$\iff A - \lambda I$ is not invertible.

$\iff |A - \lambda I| = 0$

Solving $|A - \lambda I| = 0$ for λ gives a polynomial equation ~~equal~~ in λ called the <u>characteristic polynomial</u>.

<u>Definition</u> The polynomial in λ arising from $|A - \lambda I| = 0$ is called the characteristic polynomial of A.

Example: C^∞ : space of functions that are infinitely differentiable.

$$T : C^\infty \longrightarrow C^\infty$$

$$T(f) = f'$$

Find eigenvalues / eigenvectors of T.

Find λ and f s.t. $\boxed{T(f) = \lambda f}$

$$\Rightarrow \quad f' = \lambda f \Rightarrow \frac{df}{dt} = \lambda f$$

$$\Rightarrow \int \frac{1}{f}\, df = \int \lambda\, dt \Rightarrow \ln|f| = \lambda t + C$$

$$\Rightarrow \quad K e^{\lambda t} = |f| \Rightarrow \text{All } \lambda \in \mathbb{R} \text{ are evalues}$$

Evectors are functions of the form $f \leftarrow \tilde{K} e^{\lambda t}$, $\tilde{K} \neq 0$

$\tilde{K} \lambda e^{\lambda t} = \lambda f$