MATH 430
Advanced Linear Algebra

Session 21

Dagonalization

$\dim(V) = n$

Problem : $T : V \longrightarrow V$ is linear.

To find a basis β of V such that $[T]_\beta$ is diagonal, if such a basis β exists.

Definition : T is said to be diagonalizable if there exists an ordered basis β of V such that $[T]_\beta$ is diagonal.

Let v be an eigenvector of T, i.e., $\exists$ a scalar λ s.t.

$$T(v) = \lambda v$$

β is an e-vector basis

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis of V such that each v_j is an e-vector of T.

Let λ_j be the e-value of T for v_j.

Then $T(v_j) = \lambda_j v_j$

$$T(v_1) = \lambda_1 v_1 = \lambda_1 v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$$

$$T(v_2) = \lambda_2 v_2 = 0 \cdot v_1 + \lambda_2 v_2 + \cdots + 0 v_n$$

$$\vdots$$

$$T(v_n) = \lambda_n v_n = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + \lambda_n \cdot v_n$$

$$[T]_\beta = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \longrightarrow \text{diagonal!}$$

<u>Example</u> : $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$$T(f(x)) = f(x) + (1+x) f'(x)$$

Consider the standard basis of P_2 : $\alpha = \{1, x, x^2\}$

$$T(1) = 1 + (1+x)\cdot 0 = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$

$$T(x) = x + (1+x)\cdot 1 = 1+2x = 1\cdot 1 + 2\cdot x + 0\cdot x^2$$

$$T(x^2) = x^2 + (1+x)\,2x = 3x^2+2x = 0\cdot 1 + 2\cdot x + 3\cdot x^2$$

$$[T]_\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$ Not diagonal.

E-values of T = e-values of $[T]_\alpha$.

E-values of $[T]_\alpha$ come from $\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0 \Rightarrow \underbrace{\lambda = 1, 2, 3}_{\text{Three distinct e-values.}}$$

E-vectors : $([T]_\alpha - \lambda I)x = 0$

$\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

x_1 - free

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an evector of $[T]_\alpha$.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \sim 1 + 0 \cdot x + 0 \cdot x^2 = 1$ is an evector of T.

$\lambda = 2:$ $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an e-vector of $[T]_\alpha$.

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \sim 1\cdot 1 + 1\cdot x + 0\cdot x^2 = 1+x$ is an evector of T.

$\lambda = 3:$ $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an e-vector of $[T]_\alpha$.

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \sim 1 + 2x + x^2$ is an evector of T.

Consider $\beta = \{1, 1+x, 1+2x+x^2\}$. β is a basis of P_2.

$\boxed{\beta \text{ is l.i., } \dim P_2 = 3 = \text{size}(\beta)}$

Find $[T]_\beta$.

$$T(1) = 1 = 1\cdot 1 + 0\cdot(1+x) + 0\cdot(1+2x+x^2)$$

$$T(1+x) = (1+x)+(1+x)\cdot 1 = 2(1+x) = 0\cdot 1 + 2(1+x) + 0\cdot(1+2x+x^2)$$

$$T(1+2x+x^2) = (1+2x+x^2)+(1+x)(2+2x) = 3(1+2x+x^2) = 0\cdot 1 + 0\cdot(1+x) + 3(1+2x+x^2)$$

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Diagonal – diagonal entries are the e-values.

T is diagonalizable.

Question : When can we get an eigenvector basis of V for $T : V \longrightarrow V$? If $\dim(V)$ is n and there are n distinct e-values.

Theorem : Let $\lambda_1, \dots, \lambda_k$ be distinct e-values of T. Then the corresponding e-vectors for a l.i. set.

Proof : Done by induction on k.

Theorem 5.1 : $T : V \to V$ is diagonalizable if and only if $\exists$ a set $\beta = \{v_1, \dots, v_n\}$ such that each v_i, $i = 1, \dots, n$ is an e-vector of T, and β is an ordered basis of V. In this case, the matrix $[T]_\beta$ is diagonal whose ~~entries are the~~ diagonal entries are the e-values.