MATH 430
Advanced Linear Algebra

Session 22

Diagonalization (of matrices)

$A \in M_{n \times n}$, $\quad A : \mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$
$\qquad\qquad\qquad\quad x \longmapsto Ax$

<u>Definition</u>: A matrix A is diagonalizable if $\exists$ an invertible Q such that $\underline{Q^{-1} A Q}$ is <u>diagonal</u>.

A : matrix of a linear tranformation w.r.t. the standard basis of $\mathbb{R}^n$

then $Q^{-1}AQ$: matrix of the transformation w.r.t. a new basis coming from the columns of Q.

A is diagonalizable $\Longleftrightarrow$ A has n l.i. eigenvectors.

Let $\{v_i\}_{i=1}^{n}$ be n l.i. eigenvectors of A.

Let $Q = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$ and let $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

Q^{-1} exists

$Av_i = \lambda_i v_i$

where λ_i is the eigenvalue corresponding to v_i.

$$AQ = \begin{bmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = QD$$

$AQ = QD \implies \boxed{Q^{-1}AQ = D} \implies A$ is diagonalizable

or, $A = QDQ^{-1}$

- A is ~~diagoniz~~ diagonalizable if $\exists\, Q$ (invertible) s.t. $\underbrace{Q^{-1} A Q}_{=D}$ is diagonal.
- The matrix Q is ~~the~~ a matrix whose columns are eigenvectors of A.
- The diagonal matrix D has diagonal entries that are eigenvalues of A.

Suppose A is diagonalizable. Suppose we wish to calculate A^n, n is some integer.

$$A^n = \overset{①}{(QDQ^{-1})}\overset{②}{(QDQ^{-1})} \cdots \overset{(n-1)}{(QDQ^{-1})}\overset{(n)}{(QDQ^{-1})}$$

$$= Q D^n Q^{-1}.$$ Finding D^n is simple, so, A^n is easy to calculate when A is diagonalizable.

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \quad \text{then} \quad D^n = \begin{bmatrix} \lambda_1^n & & & 0 \\ & \lambda_2^n & & \\ & & \ddots & \\ 0 & & & \lambda_n^n \end{bmatrix}$$

<u>Example</u>: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ (from last class)

Find Q and D s.t. $Q^{-1} A Q$ is diagonal.

Find A^n; $n = 100$

$\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\checkmark Q = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

is invertible because $\{v_1, v_2, v_3\}$ is a l.i. set.

$$A^{100} = Q D^{100} Q^{-1} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 3^{100} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Idea behind diagonalization :

$T : V \longrightarrow V$; $\dim(V) = n$

$A \in M_{n \times n}$

To be able to find a basis of eigenvectors of T/A. A/T is diagonalizable $\Longleftrightarrow$ A/T has n l.i. eigenvectors. One way this is possible is if A/T has n distinct evalues.

It is possible to have a diagonalizable matrix that does not have distinct e-values.

<u>Example</u> $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix (diagonalizable).

E values are $\lambda = 1, 1, 1$. $(\lambda-1)^3$: char. poly.

E vectors for $\lambda = 1$

$(I - 1 \cdot I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{(I - \lambda I)x = 0}$

x_1, x_2, x_3 are free.

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are l.i. evectors for $\lambda = 1$.

Example : $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ Is A diagonalizable?

Evalues : $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = 0$

Characteristic polynomial is $(\lambda+1)^2 = 0$.

$\lambda = \underbrace{-1, -1}_{\text{repeated}}$

To be diagonalizable, we need two l.i. e-vectors for $\lambda = -1$.

$\underset{A-\lambda I}{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Eigenvector : $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $\alpha \in \mathbb{R}$.

Not possible to find 2 l.i. e-vectors

$\Rightarrow$ A is NOT diagonalizable.