MATH 430
Advanced Linear Algebra

Session 23

Diagonalization (repeated eigenvalues)

So far: If A ($n \times n$) or $T: V \to V$ ($\dim(V) = n$) has n distinct eigenvalues then A/T is diagonalizable.

If some eigenvalues are repeated, can we still diagonalize A/T?

Yes, but not always.

Definition (1) Let λ be an e-value of A or T. Then the no. of times λ is repeated as a root of the characteristic polynomial is called the algebraic multiplicity of λ.

$0 = (\lambda-1)^3(\lambda-2)^2$ $\quad$ $\lambda = 1$ has multiplicity $= \boxed{3}$

(2) $E_\lambda :=$ eigenspace corresponding to λ
$:= \text{span}\{$all linearly independent eigenvectors for $\lambda\}$

(3) The max. number of l. i. e-vectors for $\lambda = \dim(E_\lambda)$, called the <u>geometric multiplicity</u> of λ.

<u>Theorem 5.7</u>: geometric multiplicity (= $\dim(E_\lambda)$) $\leq$ algebraic multiplicity

To be diagonalizable the <u>geometric multiplicity = algebraic multiplicity</u> for <u>each</u> eigenvalue λ.

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Example 1 $\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda = 2$ will give the third e-vector v_3

$\det(A - \lambda I) \Rightarrow \boxed{(\lambda-1)^2(\lambda-2) = 0} \Rightarrow \lambda = 1, 1, 2$

$= (1-\lambda)^2(2-\lambda)$

$\lambda = 1$ has algebraic multiplicity equal to 2.

$\lambda = 2$ " " " " 1.

Evectors for $\boxed{\lambda = 1}$: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_3 = 0$; x_1, x_2 are free

$x_1 = 1$ $\quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $x_2 = 1$ $\quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are two linearly independent e-vectors for $\lambda = 1 \Rightarrow$

geometric multiplicity $= 2$

$\lambda=2:$ $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0,\ x_2 = x_3$

x_3 - free $\quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$E_{(\lambda=2)} = \text{span}\{v_3\}$

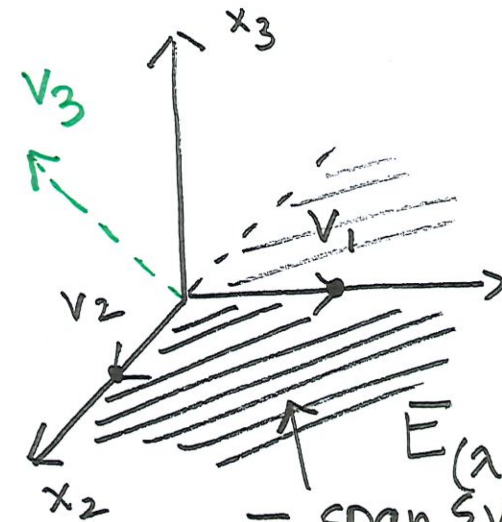
$\lambda=2$ has geometric multiplicity equal to 1.

$\{v_1, v_2, v_3\}$ is a l.i. set in $\mathbb{R}^3$; forms a basis of $\mathbb{R}^3 \Rightarrow A$ is diagonalizable.

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \lambda=1 & \lambda=1 & \lambda=2 \\ \checkmark & \checkmark & \end{array}$$

$$Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$



$E_{(\lambda=1)} = \text{span}\{v_1, v_2\}$

If $Q = \begin{bmatrix} v_3 & v_1 & v_2 \end{bmatrix}$. Then $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 2 $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ E-values are $\lambda = \underline{3, 3}, 4$

$\lambda = 3$ has algebraic multiplicity $= \boxed{2}$.

Evectors come from $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_2 = 0,\ x_3 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an evectors.

Geom. mult. $= \boxed{1 < 2}$

Impossible to find another evector that is l.i. to $v_1 \Rightarrow$ cannot find 3 l.i. evectors of A $\Rightarrow$ A is not diagonalizable.

Summary

$A \in M_{n\times n}$, $T: V \longrightarrow V$ $\quad \dim(V) = n$

1) Find all e-values of A or $[T]$.

2) Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be k distinct evalues ($k \leq n$) with algebraic multiplicities $n_1, n_2, \ldots, n_k$

$$n_1 + n_2 + \cdots + n_k = n$$

The characteristic polynomial is

$$(\lambda_1 - \lambda)^{n_1} (\lambda_2 - \lambda)^{n_2} \cdots (\lambda_k - \lambda)^{n_k}$$

3) A or $[T]$ is diagonalizable if

$$\dim(E_{\lambda_1}) = n_1$$
$$\dim(E_{\lambda_2}) = n_2$$
$$\vdots$$
$$\dim(E_{\lambda_k}) = n_k$$

(4) If $A/[T]$ is diagonalizable then $Q^{-1}AQ = \underset{D}{\text{diagonal}}$, where cols. of Q are e-vectors of A and D contains the e-values.

Caley Hamilton Theorem :

Let $A \in M_{n\times n}$ and let $f(\lambda)$ be the characteristic polynomial of A. Then

$$\boxed{f(A) = O} \to \text{zero matrix}$$

In other words, a matrix satisfies its own characteristic polynomial equation.

Ex $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ $\quad \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 5 = 0$

By Caley Hamilton, $A^2 - 2A + 5I = O$ matrix

Find $A^3 - 2A^2 + 11A$.

$$A^3 - 2A^2 + 11A = A(\underbrace{A^2 - 2A + 5I}_{=0 \text{ by Caley Hamilton}} + 6I)$$

$$= A \cdot 6I = \boxed{6A}$$