University of Idaho

MATH 430
Advanced Linear Algebra

Session 24

Inner product spaces

V - vector space on a field $\mathbb{F}\ (= \mathbb{R}, \mathbb{C})$

$\underbrace{V \times V}_{\text{Cartesian product}} = \{(x, y) : x \in V,\ y \in V\}$ is the set of ordered pairs.

Definition: An inner product is a function defined on $V \times V$ as

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}\ (= \mathbb{R}, \mathbb{C})$$
$$(x, y) \longmapsto \langle x, y \rangle$$

$\langle x, y \rangle$ → a scalar in $\mathbb{F}$

Such that

(a) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(b) $\langle cx, y \rangle = c\langle x, y \rangle$ for any $c \in \mathbb{F}$

(c) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ where the bar denotes complex conjugation.

(d) $\langle x, x \rangle > 0$ if $x \neq \vec{0}$

(a) & (b) require the inner product to be linear in the first component.

(c): If $\mathbb{F} = \mathbb{R}$ then $\langle x, y \rangle = \langle y, x \rangle$

Example 1 $V = \mathbb{R}^n$ $\quad x = (x_1, x_2, \dots, x_n)$

$y = (y_1, y_2, \dots, y_n)$

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$$ (the usual scalar product or dot product in $\mathbb{R}^n$)

This is an inner product on $\mathbb{R}^n$.

Let $n = 3$. Let $x = (1, 2, 0)$ and $y = (4, 3, 1)$

$$\langle x, y \rangle = 1(4) + 2(3) + 0(1) = 10$$

(2) $V = \mathbb{C}^n$ $\quad$ $\mathbb{C}$ - set of complex numbers.

$x = (x_1, \dots, x_n)$ $\quad$ $x_i, y_i \in \mathbb{C}$

$y = (y_1, \dots, y_n)$

$\langle x, y \rangle := \sum_{i=1}^{n} x_i \overline{y_i}$ is an inner product on $\mathbb{C}$.

$n = 2$, $V = \mathbb{C}^2$, $x = (1+i, 4)$, $y = (2-3i, 4+5i)$

$i = \sqrt{-1}$

$z = a + ib$

$a, b \in \mathbb{R}$

$\overline{z} = a - ib$

$i^2 = -1$

$$\langle x, y \rangle = (1+i)(2+3i) + 4(4-i5)$$

$$\vdots$$

$$= 15 - 15i$$

(3) V - set of all continuous functions on $\overset{a,b}{[0,1]}$

For $f, g \in V$, let

$\mathbb{F} = \mathbb{R}$

$f, g : [0,1] \to \mathbb{R}$

$$\langle f, g \rangle := \int_{0\,a}^{1\,b} f(x)\, g(x)\, dx$$

The above integral defines an inner product.

<u>Definition</u> : A vector space V with a specific inner product is called an <u>inner product space</u>, written as $(V, \langle \cdot, \cdot \rangle)$

If $\mathbb{F} = \mathbb{R}$, then it is a <u>real inner product space</u>.

If $\mathbb{F} = \mathbb{C}$, then it is a <u>complex inner product space</u>.

Theorem (Properties of inner product)

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle, \ \forall c \in \mathbb{F}$

The inner product is conjugate linear in the second component.

(c) $\langle x, \vec{0} \rangle = 0, \ \langle \vec{0}, x \rangle = 0$ for all $x \in V$.

(d) $\langle x, x \rangle = 0 \iff x = \vec{0}$

(e) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$ then $y = z$.

Proof (of (e)): Let $\langle x, y \rangle = \langle x, z \rangle$

Then $\langle x, y \rangle \overset{+(-1)}{-} \langle x, z \rangle = 0$

$\Rightarrow \langle x, y \rangle + \langle x, -z \rangle = 0$ by (b)

$\Rightarrow \langle x, y - z \rangle = 0$ by (a) for all x.

Set $x = y - z$. Then $\langle y-z, y-z \rangle = 0$.

By (d), $y - z = \vec{0} \implies y = z$. $\square$

By the first two axioms of $\langle \cdot, \cdot \rangle$ and (a) & (b) above:

$\alpha_i, \beta_j \in \mathbb{F}$
$x_i, y_j \in V$

$$\left\langle \sum_{i=1}^{n} \alpha_i x_i , \sum_{j=1}^{n} \beta_j y_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\beta}_j \langle x_i, y_j \rangle$$

$\langle x, y + \alpha z \rangle = \langle x, y \rangle + \bar{\alpha} \langle x, z \rangle$

$\langle u + cv, w \rangle = \langle u, w \rangle + c \langle v, w \rangle$

$\alpha, c \in \mathbb{F}$