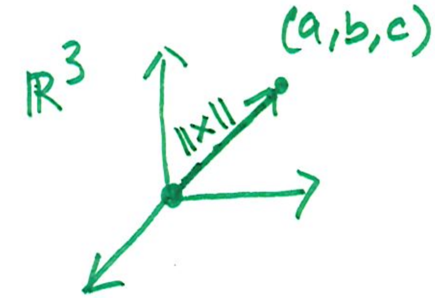
University of Idaho

MATH 430
Advanced Linear Algebra

Session 25

Inner products and norm



In $\mathbb{R}^3$, let $x = (a, b, c)$

$$\langle x, x \rangle := x \cdot x = a^2 + b^2 + c^2$$

Recall, the Euclidean length of x is

$$\|x\| = \sqrt{a^2+b^2+c^2} = \sqrt{\langle x, x \rangle}$$

Genaralize this to any inner product space.

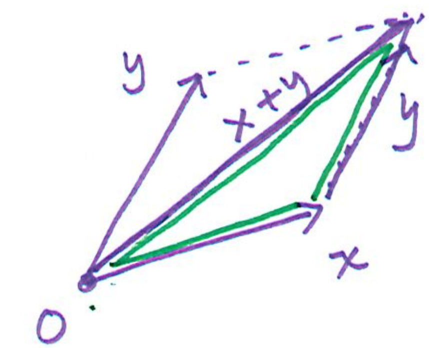
Definition: Given an inner product space V, the norm of any $x \in V$ is given by

$$\|x\| := \sqrt{\langle x, x \rangle}$$

Every inner product gives rise to a norm (length).

Distance between $\vec{x}$ and $\vec{y}$:

$$\|\vec{x}-\vec{y}\| = \sqrt{\langle \vec{x}-\vec{y}, \vec{x}-\vec{y} \rangle}$$



Properties of norm :

(a) $\|c\vec{x}\| = |c|\,\|\vec{x}\|$

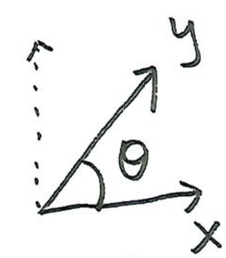
(b) $\|\vec{x}\| = 0 \iff \vec{x} = 0$

(c) $|\langle x, y\rangle| \leq \|x\|\,\|y\|$: Cauchy Schwarz Inequality

(d) $\|x+y\| \leq \|x\| + \|y\|$: Triangle Inequality

(c) In $\mathbb{R}^2$

$$\langle x, y\rangle := x \cdot y = \|x\|\,\|y\| \cos\theta$$



$|\cos\theta| \leq 1$

$$|\langle x, y\rangle| = \|x\|\,\|y\|\,|\cos\theta| \leq \|x\|\,\|y\|$$

V - inner product space

In $\mathbb{R}^2$

$\langle x, y \rangle = 0 \iff \cos\theta = 0$ or, $\theta = 90^\circ$

x & y are **perpendicular**

Definition (a) $\vec{x}$ and $\vec{y}$ in V are said to be **orthogonal** if $\langle x, y \rangle = 0$

(b) A set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is an **orthogonal set** if $\langle \vec{x}_i, \vec{x}_j \rangle = 0$, $i \neq j$.

(c) A vector $\vec{x}$ is **unit norm** if $\|\vec{x}\| = 1$.

(d) A set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is **orthonormal** if

$$\left.\begin{array}{ll} \|x_i\| = 1 & i = 1, 2, \ldots, n \\ \boxed{\langle x_i, x_j \rangle = 0} & i \neq j \end{array}\right\} \text{orthogonal + unit norm}$$

$\|x_i\|^2 = 1 \Rightarrow \langle x_i, x_i \rangle = 1$

$$\langle x_i, x_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Example (1) $\{\overset{x_1}{(1,1,0)}, \overset{x_2}{(1,-1,1)}, \overset{x_3}{(-1,1,2)}\} \subseteq \mathbb{R}^3$

is orthogonal

$\langle x_1, x_2 \rangle = 1(1) + (1)(-1) + 0(1) = 1 - 1 + 0 = 0$

$\| x_1 \| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$, $\|x_2\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$

$\|x_3\| = \sqrt{(-1)^2+1^2+2^2} = \sqrt{6}$

$\left\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2) \right\}$

is an orthonormal set.

Example (2) V - continuous functions in $[-\pi, \pi]$.

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$$\| f \| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{-\pi}^{\pi} f^2(x)\, dx}$$

$$\| 1 \| = \sqrt{\int_{-\pi}^{\pi} 1^2\, dx} = \sqrt{x \Big|_{-\pi}^{\pi}} = \sqrt{\pi - (-\pi)} = \sqrt{2\pi}$$

1 is a function

The set $\left\{ \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \dots \right\}$

is an orthonormal set.

Show that

(1) $\left\| \frac{\cos nx}{\sqrt{\pi}} \right\| = \left\| \frac{\sin nx}{\sqrt{\pi}} \right\| = 1$.

$$\left\| \frac{\cos nx}{\sqrt{\pi}} \right\| = \sqrt{\int_{-\pi}^{\pi} \frac{1}{\pi} \cos^2 nx \, dx}$$

$$\boxed{\begin{array}{l} \text{Use product to sum formulas such as} \\ \cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B) \end{array}}$$

(2) $\left\langle \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}} \right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \sin mx \, dx$

(3) $\left\langle \frac{\cos nx}{\sqrt{\pi}}, \frac{\cos mx}{\sqrt{\pi}} \right\rangle, \; n \neq m \quad = 0$

(4) $\left\langle \frac{\sin nx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}} \right\rangle, \; n \neq m \quad = 0$

Theorem: V - inner product space

Any orthogonal set of nonzero vectors in V is linearly independent.

Proof: Let $\{v_1, \cdots, v_n\}$ be orthogonal.

Let $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$

Fix j. Take inner product on both sides with v_j.

$$\langle a_1 v_1, v_j\rangle + \langle a_2 v_2, v_j\rangle + \cdots + \langle a_n v_n, v_j\rangle = \langle 0, v_j\rangle$$

$$a_1 \overset{0}{\cancel{\langle v_1, v_j\rangle}} + a_2 \overset{0}{\cancel{\langle v_2, v_j\rangle}} + \cdots + \overset{\neq 0}{a_j \cancel{\langle v_j, v_j\rangle}} + \cdots + a_n \overset{0}{\cancel{\langle v_n, v_j\rangle}} = 0$$

All the terms cancel except for the jth term:

$$a_j \langle v_j, v_j \rangle = 0$$

$$\Rightarrow \boxed{a_j = 0} \quad \text{because} \quad \langle v_j, v_j \rangle = \|v_j\|^2 \neq 0$$

Since j was arbitrary, each $a_j = 0$ and the set is l.i. $\square$