MATH 430
Advanced Linear Algebra

Session 26

Gram Schmidt orthogalization

Last class : Any orthogonal set of nonzero vectors in V is linearly independent.

Converse of the above is false.

Consider

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ in } \mathbb{R}^3.$$

This set is l. i. but is not orthogonal.

Definition : A basis that is also an orthonormal set is an orthonormal basis. (ONB)

Let $\{u_1, u_2, \cdots, u_n\}$ be an <u>ONB</u> for V.

Take $y \in V$. Then

$$\boxed{y = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n}$$

How to find $\alpha_1, \alpha_2, \cdots, \alpha_n$? In general, you may have to solve n equation in $\alpha_1, \cdots, \alpha_n$. Now, due to orthonormality, things get simpler.

$$\langle y, u_1 \rangle = \langle \alpha_1 u_1, u_1 \rangle + \langle \alpha_2 u_2, u_1 \rangle + \cdots + \langle \alpha_n u_n, u_1 \rangle$$

$$= \alpha_1 \underset{1}{\langle \cancel{u_1}, u_1 \rangle} + \alpha_2 \underset{0}{\langle \cancel{u_2}, u_1 \rangle} + \cdots + \alpha_n \underset{0}{\langle \cancel{u_n}, u_1 \rangle}$$

$$= \alpha_1 \qquad \langle u_1, u_1 \rangle = \|u_1\|^2 = 1$$

(3 (a)

$$\langle y, u_2\rangle = \langle \alpha_1 u_1, u_2\rangle + \langle \alpha_2 u_2, u_2\rangle + \cdots + \langle \alpha_n u_n, u_2\rangle$$

$$= \alpha_1 \underset{0}{\langle u_1, u_2\rangle} + \alpha_2 \underset{1}{\langle u_2, u_2\rangle} + \cdots + \alpha_n \underset{0}{\langle u_n, u_2\rangle}$$

$$= \alpha_2$$

$$\vdots$$

$$\langle y, u_n\rangle = \alpha_n$$

Each coefficient (weight) $\boxed{\alpha_j \text{ is } \langle y, u_j\rangle}$.

By an ONB, calculating the coefficients is a lot simpler, no need to solve systems of equations.

Theorem :

If $\{u_1, u_2, \dots, u_n\}$ is an ONB of V

then $y \in V$ is given by

$$y = \sum_{i=1}^{n} \underbrace{\langle y, u_i \rangle}_{\text{weight of } u_i} u_i$$

Gram Schmidt orthogonalization process

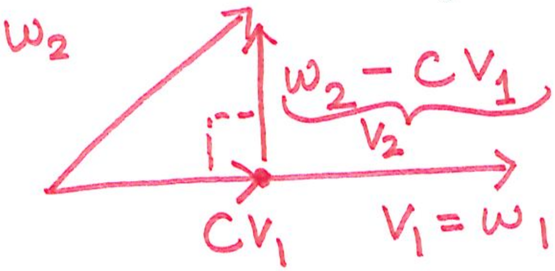
Objective : $\{w_1, w_2, \dots, w_n\}$ is linearly independent but not orthogonal

$$\Big\downarrow \text{how?}$$

$\{u_1, u_2, \dots, u_n\}$ is orthogonal or orthonormal

Think of $\mathbb{R}^2$. Consider a set of 2 vectors $\underbrace{\{w_1, w_2\}}_{\text{given}}$

$v_2 + cv_1 = w_2$



Step 1 Let $v_1 = w_1$

Step 2 Let $\boxed{v_2 = w_2 - cv_1}$

Choose c so that $\langle v_2, v_1 \rangle = 0$. by linearity

$$\boxed{0 =} \langle v_2, v_1 \rangle = \langle w_2 - cv_1, v_1 \rangle = \boxed{\langle w_2, v_1 \rangle - c \langle v_1, v_1 \rangle}$$

$$\Rightarrow \quad c = \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{\langle w_2, v_1 \rangle}{\| v_1 \|^2}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\underbrace{\boxed{\langle v_1, v_1\rangle}}_{\|v_1\|^2}} v_1 \quad \text{and} \quad \langle v_2, v_1\rangle = 0$$

$\{v_2, v_1\}$ is orthogonal.

Normalize : $u_1 = \frac{v_1}{\|v_1\|}$, $u_2 = \frac{v_2}{\|v_2\|}$

$\{u_1, u_2\}$ is orthonormal.

Extend this idea to a set of n vectors :

Given $\{w_1, w_2, \dots, w_n\}$ that forms a l. i. set.

Find an orthonormal set $\{u_1, \dots, u_n\}$.

Step 1 $\quad v_1 = w_1$ $\qquad u_1 = v_1 / \|v_1\|$

Step 2 $\quad v_2 = w_2 - \dfrac{\langle w_2, v_1 \rangle}{\| v_1 \|^2} v_1$ $\qquad u_2 = v_2 / \|v_2\|$

v_2 is orthogonal to v_1 by construction.

$$v_2 \perp v_1$$

Step 3 $\quad v_3 = w_3 - \boxed{\dfrac{\langle w_3, v_2 \rangle}{\| v_2 \|^2}} v_2 - \boxed{\dfrac{\langle w_3, v_1 \rangle}{\| v_1 \|^2}} v_1$

$u_3 = v_3 / \|v_3\|$

v_3 is orthogonal to v_1, v_2.

⋮

Step k $\quad$ We have $\{v_1, \dots, v_{k-1}\}$, an orthogonal set.

$$v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i \rangle}{\| v_i \|^2} v_i \qquad u_k = v_k / \|v_k\|$$

$\{v_1, \ldots, v_k\}$ is orthogonal.

$\{u_1, \ldots, u_k\}$ is orthonormal.

$$\text{span}\{w_1, \overset{\nearrow \text{l.i.}}{\ldots}, w_n\} = \text{span}\{v_1, \ldots, \overset{\nearrow \text{OG}}{v_n}\}$$

$$= \text{span}\{u_1, \ldots, \underset{\hookrightarrow \text{ON}}{u_n}\}$$

Spanning property is preserved.

(8)

Example $V = P_2$ $\quad \langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$

$\{\overset{w_1}{1}, \overset{w_2}{x}, \overset{w_3}{x^2}\}$ is l.i.

$$\|1\|^2 = \int_{-1}^{1} 1^2\, dx = x\Big]_{-1}^{1} = 2$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}$$

Step 1 $v_1 = 1$

Step 2

$$v_2 = x - \frac{\langle x, 1\rangle}{\|1\|^2} 1$$

$$= x - \frac{\int_{-1}^{1} x\,dx}{2} = x - \frac{\overset{0}{\cancel{x^2/2}}\Big|_{-1}^{1}}{2}$$

$$= \boxed{x}$$

Step 3

$$v_3 = x^2 - \frac{\overset{\int_{-1}^{1} x^2 dx}{\langle x^2, 1\rangle}}{\|1\|^2} 1 - \frac{\overset{\int_{-1}^{1} x^3 dx}{\langle x^2, x\rangle}}{\underset{\int_{-1}^{1} x^2 dx}{\boxed{\|x\|^2}}} x$$