MATH 430
Advanced Linear Algebra

Session 27

Orthogonal complement

<u>Definition</u> V - vector space, W is a subset of V.

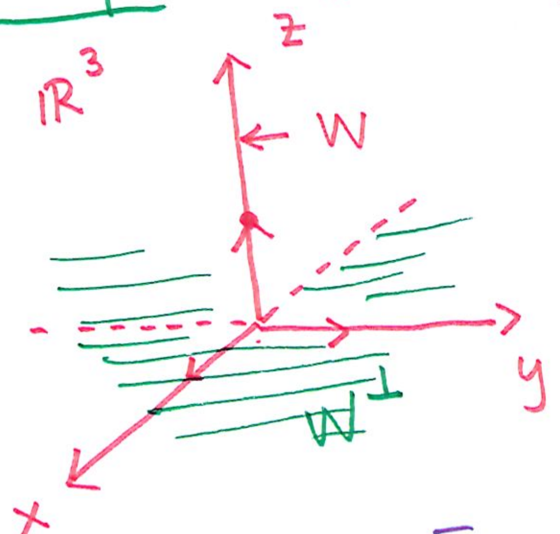
$W^{\perp}$ (say "W perp") is the set

$x \perp y = 0$

$$W^{\perp} = \{ x \in V : \langle x, y \rangle = 0 \text{ for all } y \in W \}$$

$W^{\perp}$ is called the <u>orthogonal</u> <u>complement</u> of W.

<u>Example</u> $V = \mathbb{R}^3$, $W = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ = z-axis



$W^{\perp} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ = x-y plane

$$\mathbb{R}^3 = W \oplus W^{\perp}$$

direct ~~xxxx~~ sum

For any W in V, $W^{\perp}$ is a subspace of V.

- W is a subspace. Then $W \cap W^{\perp} = \{\vec{0}\}$
- For any V, $V^{\perp} = \{\vec{0}\}$ and $\{\vec{0}\}^{\perp} = V$.

Theorem Let W be a subspace of an inner product space V. Let $y \in V$. Then

$$\boxed{\begin{array}{l} V = W \oplus W^{\perp} \\ y = u + z \end{array}}$$

$$y = u + z \qquad z = y - u$$

where $u \in W$ and $z \in W^{\perp}$; u & z are unique.
If $\{v_1, v_2, \dots, v_k\}$ is an ONB of W then

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

u : $\boxed{\text{orthogonal projection}}$ of y on W.

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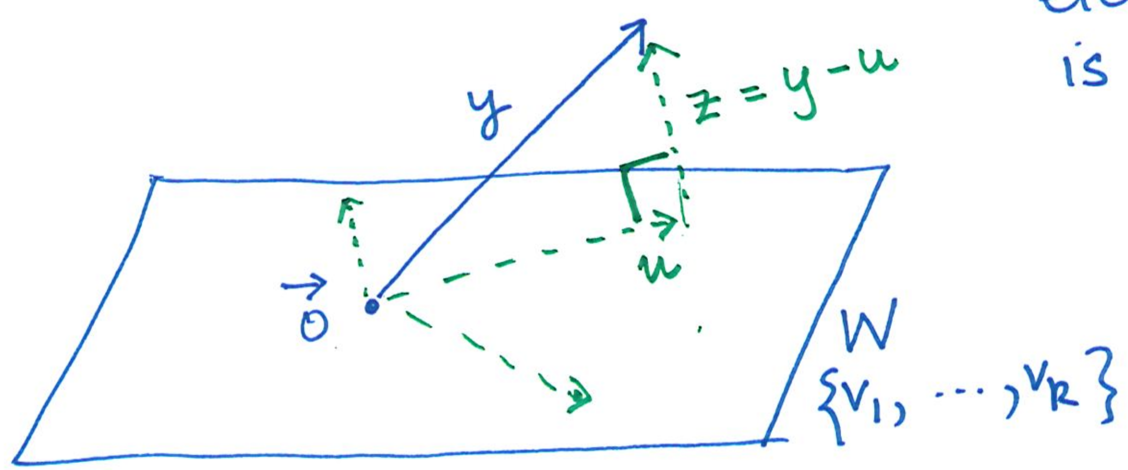
V



Problem: Find the element in W that is closest to y.

(b)

Proof. Let $\{v_1, \dots, v_k\}$ be an <u>ONB</u> of W.

$u \in W \Leftarrow u := \sum_{i=1}^{k} \langle y, v_i \rangle v_i$ Let $z = y - u$

Want to show : $z \in W^\perp$

Enough to show that $z \perp v_j$, $j = 1, \dots, k$.

$$\langle z, \underset{\substack{\uparrow \\ \text{fixed}}}{v_j} \rangle = \langle y - u, v_j \rangle = \langle y - \sum_{i=1}^{k} \overset{\text{scalar}}{\langle y, v_i \rangle} v_i, v_j \rangle$$

$$= \langle y, v_j \rangle - \sum_{i=1}^{k} \langle y, v_i \rangle \underbrace{\langle v_i, v_j \rangle}_{\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}}$$

$$= \langle y, v_j \rangle - \langle y, v_j \rangle$$

$$= 0 \quad \text{for any } j.$$

$$\Rightarrow z \in W^\perp.$$

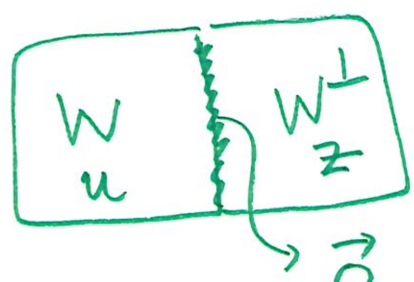
<u>Uniqueness</u> : Let $\boxed{y = u + z}$ and also $\boxed{y = \tilde{u} + \tilde{z}}$

where $u, \tilde{u} \in W$ and $z, \tilde{z} \in W^{\perp}$.

$$u + z = \tilde{u} + \tilde{z}$$

$$\Rightarrow \underbrace{u - \tilde{u}}_{\substack{\in W \\ W \text{ is a subspace}}} = \underbrace{\tilde{z} - z}_{\substack{\in W^{\perp} \\ W^{\perp} \text{ is a subspace}}} \text{ is in } W \cap W^{\perp} = \{\vec{0}\}$$

$$\left.\begin{aligned} \Rightarrow \quad u - \tilde{u} &= \vec{0} \\ \text{and} \quad \tilde{z} - z &= \vec{0} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} u &= \tilde{u} \\ z &= \tilde{z} \end{aligned}\right\} \Rightarrow u, z \text{ are unique.}$$

$V =$ 

$u + z = y \in V$

$\square$

Example $V = P_3$ $\quad \langle f, g\rangle = \int_{-1}^{1} f(x)\,g(x)\,dx$

$W = P_2$, $y = x^3 \in P_3$ $\quad \{1, x, x^2\}$ Gram Schmidt

ONB for P_2 : $\left\{ \underbrace{\frac{1}{\sqrt{2}}}_{u_1}, \underbrace{\sqrt{\frac{3}{2}}\,x}_{u_2}, \underbrace{\sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)}_{u_3} \right\}$

Orthogonal projection of y on $W = P_2$ is

$$u = \langle y, u_1\rangle u_1 + \langle y, u_2\rangle u_2 + \langle y, u_3\rangle u_3$$

$$= \underbrace{\left\langle x^3, \frac{1}{\sqrt{2}}\right\rangle}_{0}\frac{1}{\sqrt{2}} + \underbrace{\left\langle x^3, \sqrt{\frac{3}{2}}\,x\right\rangle\sqrt{\frac{3}{2}}\,x}_{\vdots} + \underbrace{\left\langle x^3, \sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)\right\rangle}_{0}\sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)$$

$$u = 3/5\,x$$

$3/5\,x$ is closest to $y = x^3$ in P_2.

Corollary: V - vector space with subspace W. Let $\{v_1, \dots, v_k\}$ be an ONB of W. Then

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

orthogonal proj. of y

is the unique vector in W that is closest to $y \in V$:

$$\| u - y \| \leq \| x - y \| \quad \text{for any } x \in W.$$

Proof: $y = u + z$, $u \in W$, $z \in W^{\perp}$

Take $x \in W$. $u - x \in W$ because W is a subspace.

$$\|y-x\|^2 = \|u+z-x\|^2 = \|u-x+z\|^2$$

$$= \langle u-x+z, \; u-x+z \rangle$$

$$= \langle u-x, u-x \rangle + \cancelto{0}{\langle u-x, z \rangle} + \cancelto{0}{\langle z, u-x \rangle} + \langle z, z \rangle$$

$$= \|u-x\|^2 + \|z\|^2$$

$$\geq \|z\|^2 = \|y-u\|^2$$

$\square$