MATH 430
Advanced Linear Algebra

Session 28

Adjoint of a linear operator (6.3)

transformation

Example $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ $\quad T : \overset{\mathbb{R}^n}{\mathbb{R}^2} \longrightarrow \overset{\mathbb{R}^m}{\mathbb{R}^2}$

$x = (x_1, x_2) \longmapsto A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \end{bmatrix}$

$T(x) := Ax$. $\quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$

$\langle T(x), y \rangle = \langle Ax, y \rangle = \boxed{(2x_1 + 3x_2)y_1 + (x_1 - x_2)y_2}$

$A^T = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \qquad A^T y = \begin{bmatrix} 2y_1 + y_2 \\ 3y_1 - y_2 \end{bmatrix}$

$\boxed{\langle x, A^T y \rangle} = \boxed{x_1(2y_1 + y_2) + x_2(3y_1 - y_2)} = \langle T(x), y \rangle$

$\langle Ax, y \rangle = \langle x, A^T y \rangle$: true for any real matrix

A^T : adjoint of A if A is real.

If A is a complex-valued matrix, A^* is defined as its complex conjugate transpose, and then we get

$$\boxed{\langle Ax, y \rangle = \langle x, A^* y \rangle}$$: true for any matrix

$$A = \begin{bmatrix} 1-2i & 3 \\ 1 & i \end{bmatrix} \qquad A^* = \begin{bmatrix} 1+2i & 1 \\ 3 & -i \end{bmatrix}$$

(See Example 2 in 6.3)

Note : $A^* = A^T$ when A is real

A^* : adjoint of the matrix A.

$V \longrightarrow V$

Definition: Let $T : V \longrightarrow W$ be linear.

Let $\dim(V) < \infty$, and $\dim(W) < \infty$.

The adjoint of T is the unique transformation

$V \longrightarrow V$

$T^* : W \longrightarrow V$ such that

$$\boxed{\langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V}$$

In the finite dimensional setting, the adjoint always exists. T^* is linear.

Thm 6.10: $T : V \longrightarrow V$, let β be an ONB of V.
Assume $\dim(V) < \infty$. Then

$$[T^*]_\beta = [T]_\beta^*$$

Example V - set of all sequences

$T(y) = \{y_2, y_3, \ldots\}$

$x \in V$; $x = \{x_1, x_2, x_3, \ldots\}$

Inner product is the dot product

S: right shift operator

T: left shift operator

$S: V \longrightarrow V$, $S(x) := \{0, x_1, x_2, x_3, \ldots\}$

Let $y = \{y_1, y_2, y_3 \ldots\}$

Find S^*.

$$\langle S(x), y \rangle = 0y_1 + x_1y_2 + x_2y_3 + x_3y_4 + \cdots$$

by defn. $\|$

$$= x_1y_2 + x_2y_3 + x_3y_4 + \cdots$$

$$\langle x, S^*(y) \rangle \qquad = \{x_1, x_2, x_3, \ldots\} \cdot \{y_2, y_3, y_4, \ldots\}$$

same

$$= \langle x, T(y) \rangle$$

$$\Rightarrow S^* = T$$

S^* is

$S^*(x) = \{x_2, x_3, \ldots\}$

Properties of the adjoint T^*:

By definition: $\langle T(x), y\rangle = \langle x, T^* y\rangle$

$\boxed{T: V \to V}$

$$\langle x, T(y)\rangle \underset{\substack{\uparrow \\ \text{by properties} \\ \text{of } \langle \cdot,\cdot\rangle}}{=} \overline{\langle T(y), x\rangle} \underset{\substack{\uparrow \\ \text{defn. of adjoint}}}{=} \overline{\langle y, T^*(x)\rangle} = \langle T^*(x), y\rangle$$

(1) $\langle T^*(x), y\rangle = \boxed{\langle x, T(y)\rangle}$

(2) Also, $\langle T^*(x), y\rangle \underset{\substack{\uparrow \\ \text{definition of} \\ \text{adjoint}}}{=} \boxed{\langle x, (T^*)^*(y)\rangle}$

Comparing (1) & (2) gives

(i) $(T^*)^* = T$

(ii) $T, U: V \to V$, $(T+U)^* = T^* + U^*$

(iii) $(cT)^* = \bar{c}\,T^*$; c is a scalar

(iv) $(TU)^* = U^* T^*$

Replacing T and U by matrices will still make the properties (i) — (iv) hold.

(v) T^* is linear. This means that

$$T^*(\overset{cy_1 + y_2}{ax + y}) = \overset{c}{a}\, T^*(\overset{y_1}{x}) + T^*(\overset{y_2}{y})$$

$$\langle x, T^*(cy_1 + y_2)\rangle \underset{\text{by linearity}}{=} \langle T(x), cy_1 + y_2\rangle$$

$$= \bar{c}\,\langle T(x), y_1\rangle + \langle T(x), y_2\rangle$$

$$= \bar{c}\,\langle x, T^*(y_1)\rangle + \langle x, T^*(y_2)\rangle$$

$$\langle x, cT^*(y_1) + T^*(y_2)\rangle \quad \underset{\text{by linearity}}{=} \langle x, c\,T^*(y_1) + T^*(y_2)\rangle$$