MATH 430
Advanced Linear Algebra

Session 29

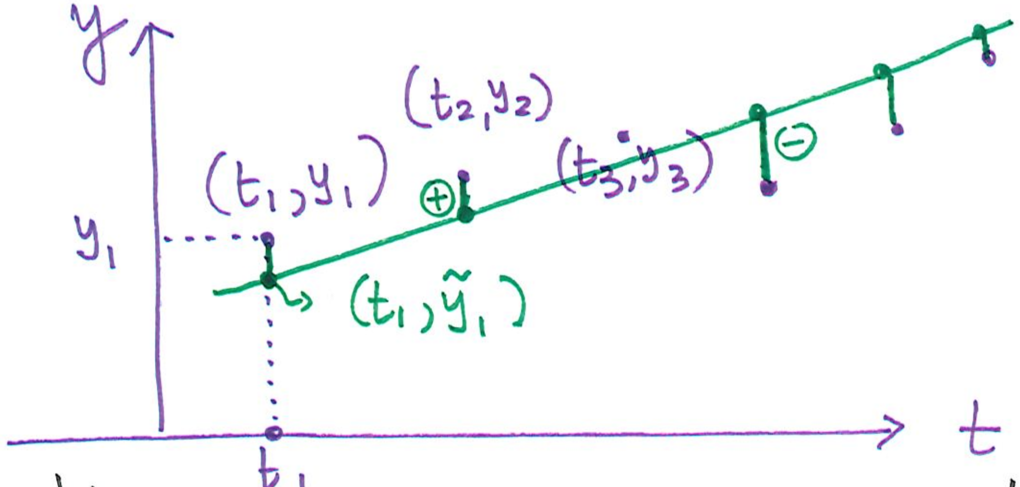
University of Idaho

Least Squares Approximation
(an application of adjoint)

Given some data set :

$(t_1, y_1), (t_2, y_2), \ldots, (t_m, y_m)$

t_i - ith time point , y_i - value at t_i



Want to fit a curve (say, a straight line) through the points

quadratic curve / ~~straight line~~ means $\tilde{y} = ct^2 + dt + e$ / $\tilde{y} = ct + d$

Want to find c, d so that the error is minimized, i.e., $\| y - \tilde{y} \|$ is minimized.

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$$\begin{array}{c} \\ \tilde{y}_1 = \\ \tilde{y}_2 = \\ \vdots \\ \tilde{y}_m = \end{array} \begin{bmatrix} \overset{ct_1^2+dt+e}{ct_1 + d} \\ ct_2 + d \\ \vdots \\ ct_m + d \end{bmatrix} = \overset{A}{\begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_m & 1 \end{bmatrix}} \overset{x}{\begin{bmatrix} c \\ d \end{bmatrix}} \quad \begin{matrix} c \\ d \\ e \end{matrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$t_m^2 \quad t_m \quad 1$

x - unknown

$$\text{Error} := \sum_{i=1}^{m} (y_i - \tilde{y}_i)^2 = \sum_{i=1}^{m} (y_i - ct_i - d)^2$$

$$= \langle y - Ax, y - Ax \rangle$$

$$= \| y - Ax \|^2.$$

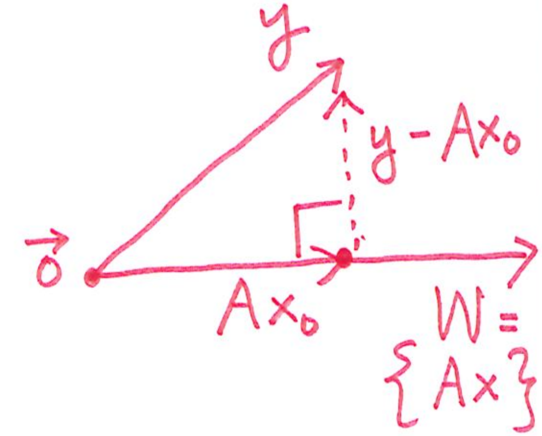
Find $x = \begin{bmatrix} c \\ d \end{bmatrix}$ so that the error is minimized, i.e. $\| y - Ax \|^2$ is minimized.

In the above setting $A \in M_{m \times 2}$

If we want to fit a curve of degree $n-1$, then $A \in M_{m \times n}$, $x \in \mathbb{F}^n$, $Ax \in \mathbb{F}^m$, $y \in \mathbb{F}^m$.

Problem [Theorem 6.12, Sec 6.3] :
Find $\underline{x_0} \in \mathbb{F}^n$ such that given $y \in \mathbb{F}^m$ and $A \in M_{m\times n}$, $\| y - Ax_0 \|$ is minimum, i.e.

$$\| y - Ax_0 \| \le \| y - Ax \| \quad \forall x \in \mathbb{F}^n .$$



<u>Solution</u> : $y \in V = \mathbb{F}^m$ is given.
$W = \{ Ax : x \in \mathbb{F}^n \}$ for a given A in $M_{m\times n}$. $W = Col(A)$.

$Ax_0 \in W$ must be "closest" to y.
If $y - Ax_0$ is orthogonal to W, then

$$\| y - Ax_0 \| \le \| y - Ax \| , \forall x \in \mathbb{F}^n$$

This mean

$$\langle A\overset{\in W}{x}, y - Ax_0 \rangle = 0 \quad \text{for all } x \in \mathbb{F}^n$$

$$\Rightarrow \langle x, A^*(y - Ax_0) \rangle = 0 \quad "$$

↑ definition of A^*

$$\Rightarrow A^*(y - Ax_0) = \vec{0}$$

$$\Rightarrow A^* y = A^* A x_0$$

The x_0 that satisfies $\boxed{A^* A x_0 = A^* y}$ solves the minimization problem.

$$\left[A^*_{n\times m} A_{m\times n} \right]_{n\times n}$$

Lemma ~~Note~~ : rank (A^*A) = rank(A) (Lemma 2, Sec. 6.3)

Show that null spaces of A^*A and A are the same

Then use Dimension Thm.

Due to the above lemma, if rank$(A) = n$ then rank$(A^*A) = n$, and since A^*A is $n \times n$, A^*A will then be invertible.

Suppose rank$(A) = n$, then $(A^*A)^{-1}$ exists and the solution is

$$\boxed{x_0 = (A^*A)^{-1} A^* y}$$

Least squares solution.

Example

t	1	2	3	4	5
y	2	3	5	7	?

$$y = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

(1) Linear fit : $\tilde{y} = ct + d$

$$x = \begin{bmatrix} c \\ d \end{bmatrix} \qquad A = \begin{array}{c} \begin{array}{cc} t & \end{array} \\ \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \end{array} \qquad \begin{bmatrix} c \\ d \end{bmatrix} = (A^*A)^{-1}A^*y$$

$d = 0, \quad c = 1.7$

Model:

$\tilde{y} = 1.7\, t \implies$ At $t = 5$, $\tilde{y} = 1.7 \times 5$

(2) Quadratic fit : $\tilde{y} = ct^2 + dt + e$

$$\begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_m \end{bmatrix} = \begin{bmatrix} ct_1^2 + dt_1 + e \\ \vdots \\ ct_m^2 + dt_m + e \end{bmatrix} = \underbrace{\begin{bmatrix} t_1^2 & t_1 & 1 \\ \vdots & \vdots & \vdots \\ t_m^2 & t_m & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c \\ d \\ e \end{bmatrix}}_{x}$$

$$\begin{bmatrix} c \\ d \\ e \end{bmatrix} = (A^*A)^{-1}A^*y$$

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For 2

$$A = \begin{array}{c} \begin{array}{ccc} t^2 & t & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \end{array}$$