MATH 430
Advanced Linear Algebra

Session 3

Subspaces (1.3)

A subset W of a vector space V (over a field $\mathbb{F}$) is called a <u>subspace</u> of V if W is a vector space in its own right.

W is a subspace of V if and only if the following three conditions hold

(a) $\vec{0} \in W$

(b) $x \oplus y \in W$ whenever $x, y \in W$ [closed under addition]

(c) For every $c \in \mathbb{F}$ and $x \in W$, $cx \in W$ [closed under scalar multiplication]

Note : The field $\mathbb{F}$ is same for both V & W.
The zero vector $\vec{0}$ is the same for both V & W.

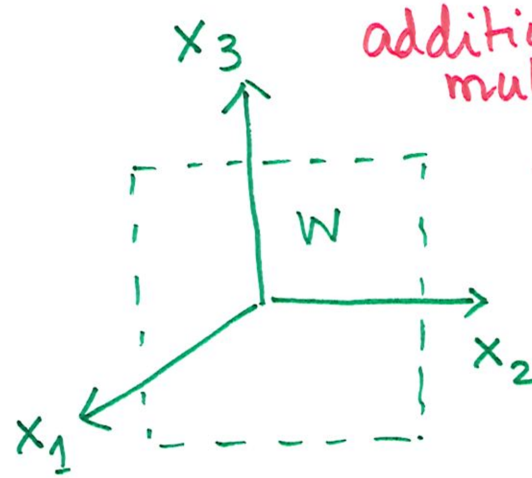
Examples (1) V & $\{\vec{0}\}$ are subspaces of V.

2 a $V = \mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$ $\mathbb{F} = \mathbb{R}$

use componentwise addition & scalar multiplication

$W = \{(x_1, x_2, x_3) : x_1 = 0\}$

$= x_2 \text{-} x_3$ plane



$\vec{0} = (0,0,0)$

Show that W is a subspace.

(a) $\vec{0} = (\overset{x_1}{0}, \overset{x_2}{0}, \overset{x_3}{0}) \in W$ because for $\vec{0}$, $x_1 = 0$

(b) Take $a = (0, a_2, a_3)$ and $b = (0, b_2, b_3)$ in W

Then $a + b = (0, a_2 + b_2, a_3 + b_3) \in W$

(c) Let $c \in \mathbb{F}$, and $a = (0, a_2, a_3) \in W$.

Then $ca = (c0, ca_2, ca_3) = (0, ca_2, ca_3)$ and is in W.

2(b) $W = \{x_1, x_2, x_3) : x_2 = 0\}$ is also a Subspace of $V = \mathbb{R}^3$ $\rightarrow$ $x_1 - x_3$ plane

3 $V = M_{m \times n}(\mathbb{R})$,

W = $m \times n$ matrices with <u>nonnegative entries</u>

W is <u>NOT</u> a subspace of V.

Let $A \in W$ (i.e. A has all nonnegative entries)

Take $c \in \mathbb{R}$, $c < 0$. Then $cA \notin W$ since cA will have negative entries.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $c = -1$, $cA = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \notin W$

$m = n = 2$

4 $V = \mathbb{R}^3$, $Y = \{(x_1, x_2, x_3 : x_2 = 0\}$

$Z = \{(x_1, x_2, x_3 : x_1 = 0\}$

We know that Y, Z are both subspaces of V.

Is $Y \cup Z$ also a subspace?

$y = (1, 0, 3) \in Y$, $z = (0, 2, 5) \in Z$

$y + z = (1, 2, 8) \notin Y, Z \Rightarrow y + z \notin Y \cup Z$

NOT 0

NOT 0

Thus $Y \cup Z$ is not a subspace.

Conclusion: The union of subspaces need not be a subspace.

<u>Theorem</u>: If Y, Z are subspaces of V, then $Y \cap Z$ is also a subspace of V.

<u>Proof</u> (a) $\underbrace{\vec{0} \in Y \text{ and } \vec{0} \in Z}_{\text{by assumption}}$

Thus $\vec{0} \in Y \cap Z$.

(b) Take $a, b \in Y \cap Z$.

$a, b \in Y \Rightarrow a+b \in Y$ [because Y is a subspace]

$a, b \in Z \Rightarrow a+b \in Z$ [Z is a subspace]

$\Rightarrow a+b \in Y \cap Z$

(c) Take $x \in Y \cap Z$ and $c \in \mathbb{F}$.

$$\left.\begin{aligned} x \in Y &\Rightarrow \underline{cx \in Y} \\ x \in Z &\Rightarrow \underline{cx \in Z} \end{aligned}\right\} \text{ Y \& Z are both subspaces}$$

$\Rightarrow cx \in Y \cap Z$

Therefore, $Y \cap Z$ is a subspace of V. $\square$

<u>Remark</u>: In general if $Y_1, \ldots, Y_n$ are subspaces of V then $Y_1 \cap Y_2 \cap \cdots \cap Y_n$ is a subspace of V. This is a way of forming new subspaces from existing ones.