MATH 430
Advanced Linear Algebra

Session 30

Normal & self-adjoint operators (6.4)

V- vector space $T : V \longrightarrow V$ is linear

T is diagonalizable $\Longleftrightarrow$ V has a basis of evectors of T.

We are interested in T so that V has an orthonormal basis (ONB) of eigenvectors of T.

If V is a complex inner product space ~~where~~ and $T : V \longrightarrow V$ then

- T is normal $\Longleftrightarrow$ V has an ONB of e-vectors of T.

If V is a real inner product space then

- T is self adjoint $\Longleftrightarrow$ V has an ONB of e-vectors of T.

Recall: $T : V \longrightarrow W$ is linear.

Thm 6.8, Thm 6.9 in 6.3

$\dim(V) < \infty$, $\dim(W) < \infty$

$T^* : W \longrightarrow V$ s.t. $\langle T(x), y \rangle = \langle x, T^* y \rangle$

- T^* always exists in the finite dimensional setting

$\langle x, T(y) \rangle = \langle T^*(x), y \rangle$

- Now we consider $W = V$, so

$$T : V \longrightarrow V$$

Definition: Let V be an inner product space and $T : V \longrightarrow V$ be linear. T is normal if

$$T \cdot T^* = T^* \cdot T$$

For a matrix A, it is ~~adjoint~~ normal if $A^* A = A A^*$.

Example Recall the rotation matrix in $\mathbb{R}^2$:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ rotates a vector counterclockwise by θ

$$AA^* = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

use $\sin^2 + \cos^2 = 1$

$$A^*A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Thus A is normal.

V : an inner product space

Properties of normal operators $T: V \to V$

(a) If T is normal then $\|T(x)\| = \|T^*(x)\|$ for all $x \in V$.

$T^*T = TT^*$

Proof : $\|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle T^*T(x), x \rangle$

$\underset{\substack{\uparrow \\ T \text{ is normal}}}{=} \langle TT^*(x), x \rangle = \langle T^*(x), T^*(x) \rangle$

$= \|T^*(x)\|^2$

$\Rightarrow \|T(x)\| = \|T^*(x)\|$

(b) If T is normal then $T - cI$ is normal for any scalar c.

Proof: Want to show $(T-cI)(T-cI)^* = (T-cI)^*(T-cI)$

$$(T-cI)(T-cI)^* = (T-cI)(T^* - \bar{c}I)$$

$$= TT^* - \bar{c}T - cT^* + c\bar{c}I$$

T is normal

$$= T^*T - \overset{I}{c}T^* - \bar{c}T - c\bar{c}I$$

$$= T^*(T-cI) - \bar{c}(T-cI)$$

$$= (T^* - \bar{c}I)(T-cI)$$

$$= (T-cI)^*(T-cI)$$

$T: V \rightarrow V$

(c) If T is normal and if x is an e-vector of T, then x is also an e-vector of T^*.

Proof: Let $\boxed{x}$ be an e-vector of T. Let λ be the corresponding e-value.

$$T(x) = \lambda x$$

Consider $U = T - \lambda I$. By (b), U is normal.

$$U(x) = T(x) - \lambda x = \lambda x - \lambda x = \vec{0} \Rightarrow \|U(x)\| = 0$$

$\|\vec{v}\| = 0 \Leftrightarrow v = \vec{0}$

By (a) $\|U^*(x)\| = \|U(x)\| = 0$

$$\Rightarrow \|(T - \lambda I)^*(x)\| = 0 \Rightarrow \|(T^* - \bar{\lambda} I)x\| = 0$$

$$\Rightarrow (T^* - \bar{\lambda} I)x = \vec{0} \Rightarrow T^* x = \bar{\lambda} x$$

$\Rightarrow x$ is an evector of T^*

$\bar{\lambda}$ is the corresponding e-value of T^*

(d) If x_1, x_2 are eigenvectors of T corresponding to <u>distinct</u> eigenvalues then they are <u>orthogonal</u> provided that T is normal.

<u>Proof</u>: Want to show: $\langle x_1, x_2 \rangle = 0$.

Let λ_1 be the e-value of x_1, λ_2 be the e-value of x_2. $\boxed{\lambda_1 \neq \lambda_2}$

$$\lambda_1 \langle x_1, x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle = \langle T(x_1), x_2 \rangle$$

$$= \langle x_1, T^*(x_2) \rangle \underset{\text{by (c)}}{=} \langle x_1, \bar{\lambda}_2 x_2 \rangle$$

$$= \lambda_2 \langle x_1, x_2 \rangle \Rightarrow (\lambda_1 - \lambda_2) \langle x_1, x_2 \rangle = 0$$

$$\Rightarrow \langle x_1, x_2 \rangle = 0$$