MATH 430
Advanced Linear Algebra

Session 31

Normal & self-adjoint operators

$T: V \to V$, linear
$\dim(V) = n$

$T^* \circ T = T \circ T^*$: normal

When would T be normal?

Suppose that T has an orthonormal eigenvector basis V. Call this basis β. Then

$$[T]_\beta = D \text{ (diagonal)}$$

$$[T^*]_\beta = [T]_\beta^* \quad (\beta \text{ is orthonormal})$$
$$= D^* \text{ (diagonal)}$$

$$D = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{bmatrix}$$

$$D^* = \begin{bmatrix} \bar{d}_1 & & 0 \\ & \bar{d}_2 & \\ 0 & & \ddots \; \bar{d}_n \end{bmatrix}$$

$$d_i \bar{d}_i = |d_i|^2$$

$[TT^*] = DD^*$ $\quad$ $[T^*T] = D^*D$

$DD^* = D^*D$ and so $TT^* = T^*T$

D is diagonal $\qquad$ T is normal

(1) If T has an orthonormal set of eigenvectors that form a basis of V then T is normal.

The converse of (1) is true only if V is a complex inner product space.

$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Rotation matrix

Example $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ A is normal

$A^* = \begin{bmatrix} \cos & \sin \\ -\sin & \cos \end{bmatrix} \neq A$

$i = \sqrt{-1}$

E-values $\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0 \quad \ldots$

$\lambda = \cos\theta \pm i\sin\theta$ $e^{\pm i\theta}$

$v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ are eigenvectors.

$\{v_1, v_2\}$ cannot form a basis of $\mathbb{R}^2 = V$.

Being normal does not guarantee having an ONB of evectors unless V is a complex inner product space.

<u>Theorem</u>: Let $T : V \to V$, <u>V is finite dimensional</u>, and V is a <u>complex inner product</u> space. Then T is normal if and only if there exists an orthonormal basis of V consisting of eigenvectors of T.

Note: The theorem does not hold if V is infinite dimensional. Look at Example 3 in Sec. 6.4

<u>Definition</u>: Let $A \in M_{n\times n}$. A is <u>self-adjoint</u> (Hermitian) if $A^* = A$. (If A is real then this means that $A^T = A$, i.e., A is symmetric)

If $T: V \longrightarrow V$ (T linear) then T is self-adjoint if $T^* = T$.

(Recall: $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$)

<u>Examples</u>: $A = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix}$, $A^* = \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix} = A^T = A$

A is self-adjoint or symmetric.

$A = \begin{bmatrix} 1 & 2-i \\ 2+i & 4 \end{bmatrix}$ $A^* = \begin{bmatrix} 1 & 2-i \\ 2+i & 4 \end{bmatrix} = A$

A is self-adjoint.

Properties of self adjoint operators

(a) self-adjoint operators are normal.

Proof

same!

$$A^* A = AA = A^2 \quad (\text{since } A^* = A)$$

$$A A^* = AA = A^2$$

(b) Eigenvectors of distinct eigenvalues are orthogonal.

Since self-adjoint operators are normal, this holds due a property of e-vectors of normal operators.

(c) Self-adjoint operators have real e-values.

Proof: Let λ be an eigenvalue of T where T is self adjoint.

$\boxed{T(x) = \lambda x}$, $\vec{x} \neq \vec{0}$

T is also normal : $T^*(x) = \bar{\lambda} x$

T is self adjoint : $\boxed{T(x) = \bar{\lambda} x}$ (replace T^* by T above

Thus $\lambda = \bar{\lambda} \Rightarrow \lambda$ is real. $\square$

(Note that the rotation matrix is normal but not self adjoint.

<u>Theorem</u> : Let $T : V \longrightarrow V$, V is a real inner product space. T is <u>self adjoint</u> if and only if T has an orthonormal set of eigenvectors that forms a basis for V.