University of Idaho

MATH 430
Advanced Linear Algebra

Session 32

University of Idaho

Self-adjoint operators (review)

Theorem Let $T: V \to V$, V is a <u>real</u> inner product space. T is self-adjoint $\iff$ T has an orthonormal set of eigenvectors that forms a basis for V.

Example $V = \mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$A^* = A$, A is self-adjoint.

E-values : $\lambda = 1, 1, 1, -1$

$\lambda = 1$: $(A - 1I)\vec{x} = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = x_3$ $\quad$ x_1, x_2, x_4 are free

$$\underset{v_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}0\\1\\1\\0\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}$$

$\{v_1, v_2, v_3\}$ is orthogonal.

l.i.

$\downarrow$ G-S

ON

$\lambda = -1$

$(A + 1I)x = 0$

$$\begin{bmatrix}2 & 0 & 0 & 0\\0 & 1 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$x_1 = x_4 = 0$

$x_2 = -x_3$

One free var.

$$\underset{v_4}{\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}}$$

$\{v_1, v_2, v_3, v_4\}$ is orthogonal.

Normalize : $\left\{ \underset{u_1}{\overset{\lambda=1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}}, \underset{u_2}{\overset{\lambda=1}{\begin{bmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}}}, \underset{u_3}{\overset{\lambda=1}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}}, \underset{u_4}{\overset{\lambda=-1}{\begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}}} \right\}$

is an <u>orthonormal set</u>* of eigenvectors of A. This forms a basis of $\mathbb{R}^4$.

l.i. set

$$U = \begin{bmatrix} \overbrace{\begin{matrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{matrix}}^{\text{must be normalized}} \end{bmatrix}$$

$\mathcal{U}$ is <u>unitary/orthogonal</u>.

$$A \sim D$$

$$\mathcal{U}^{-1} = \mathcal{U}^*$$

$$\mathcal{U}^{-1} A \, \mathcal{U} = D$$

Self-adjoint (A) diagonal (D).

$$D = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}$$

Unitary / Orthogonal matrices and operators

Definition: A matrix Q is said to be orthogonal or unitary if

$$Q^*Q \underset{\substack{\uparrow \\ \text{normal}}}{=} QQ^* = I$$

Unitary: Q has complex entries

Orthogonal: Q " real entries.

Example: The rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A$

satisfies $A^*A = AA^* = I$.

Properties: (1) Orthogonal/unitary matrices are normal.

(2) For unitary/orthogonal

$$Q^{-1} = Q^*$$

Finding the inverse is very easy!

(3) The columns (rows) of a $\overset{\text{OG}}{\text{unitary}}$ matrix form a orthonormal set of vectors.

Let Q be $n \times n$ and unitary. Let $v_1, \dots, v_n$ be the columns of Q.

$$\underset{\underset{I\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}}{\shortparallel}}{Q^* Q} = \begin{bmatrix} - & \bar{v}_1 & - \\ - & \bar{v}_2 & - \\ & \vdots & \\ - & \bar{v}_n & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \overset{\overset{I}{\shortparallel}}{\begin{bmatrix} \langle v_1, v_1\rangle^{1} & \langle v_2, v_1\rangle^{0} & \cdots & \langle v_n, v_1\rangle^{0} \\ \langle v_1, v_2\rangle^{0} & \langle v_2, v_2\rangle^{1} & \cdots & \langle v_n, v_2\rangle \\ \vdots & & & \\ \langle v_1, v_n\rangle & \langle v_2, v_n\rangle & \cdots & \langle v_n, v_n\rangle^{1} \end{bmatrix}}$$

$$\text{ON} \Leftarrow \begin{cases} \underbrace{\langle v_i, v_i\rangle}_{=\|v_i\|^2} = 1 & \forall i \\ \langle v_i, v_j\rangle = 0 & i \neq j \end{cases}$$

The converse of 3 is true. If $\{v_1, \cdots, v_n\}$ is ON then $Q = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ is unitary.

If Q is unitary

$QQ^* = I$: The rows must be ON.

$$Q = \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

Caution: The rows/columns of an orthogonal matrix must have unit norm.
Do not confuse with orthonormal set.

From the example we did today:

- A (in that example) is said to be diagonalizable via a unitary matrix.

- A is said to be unitarily equivalent to a diagonal matrix D.

Definition : If A and B are two matrices such that $A = Q B Q^*$ for some unitary matrix Q then A and B are said to be unitarily equivalent.

Thm 6.20

A is $\overbrace{\text{real symmetric}}^{\text{self-adjoint}}$ matrix

$\iff$ A is $\underbrace{\text{orthogonally}}$ equivalent to a diagonal matrix

Thm 6.19 A is complex normal matrix

$\iff$ A is unitarily equivalent to a diagonal matrix.