MATH 430
Advanced Linear Algebra

Session 33

8(b) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ self adjoint. (symmetric, real)

We should be able to find orthogonal P s.t. $P^{-1} = P^*$

$P^* A P = D$ $\qquad$ $P \underset{\substack{\downarrow \\ \text{diagonal}}}{D} P^* = A$

Step 1 Find e-values : $\lambda = \underbrace{1, 1}_{\text{repeated}}, 4$

Step 2 : Find orthonormal eigenvectors.

$\lambda = 1$: $(A - I)\vec{x} = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

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$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$

2 free variables : x_2, x_3

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ \overset{y_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \overset{y_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$

$x_2 = 1, x_3 = 0$; $x_2 = 0, x_3 = 1$

$= \dim(\mathrm{Nul}(A-I))$

$= \dim(E_{\lambda=1})$

$\boxed{\langle \overset{v_1}{y_1}, \overset{y_2}{y_2} \rangle = 1 \neq 0 \quad \{y_1, y_2\} \text{ is NOT orthogonal.}}$

$\{y_1, y_2\}$ is l.i. $\xrightarrow{GS}$ $\{u_1, u_2\}$ is ON

Recall $\mathrm{span}\{y_1, y_2\} = \mathrm{span}\{u_1, u_2\}$

$v_1 = y_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \|v_1\| = \sqrt{2}$

$v_2 = y_2 - \dfrac{\langle v_1, y_2 \rangle}{\|v_1\|^2} v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$\|v_2\| = \sqrt{3/2}$

$$\underbrace{\left\{ \underset{u_1}{\frac{v_1}{\|v_1\|}}, \underset{u_2}{\frac{v_2}{\|v_2\|}} \right\} = \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}}_{\text{ON}} \quad \text{for } \lambda = 1$$

$$\sqrt{\frac{2}{3}}\,\frac{1}{2} = \frac{1}{\sqrt{3}\sqrt{2}} = \frac{1}{\sqrt{6}}$$

$\lambda = 4$ $\quad (A - 4I)\vec{x} = 0$

$$\therefore \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = x_2 = x_3$$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an evector, norm is $\sqrt{3}$.

Normalize : $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ \underbrace{0 \quad\quad \sqrt{2/3}}_{\lambda = 1} & & \underbrace{1/\sqrt{3}}_{\lambda = 4} \end{bmatrix}$$

orthogonal matrix $\leftarrow$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A = P \overset{D}{\cancel{\ }} P^*$$

$$P^{-1} = P^*$$

7(b) $V = M_{2\times 2}$

$T(A) = A^T$

$$T : V \longrightarrow V$$
$$A \longmapsto A^T$$

T : SA ✓ ? normal ✓ ?

<u>Step 1</u> : Find $[T]$.

Start with a basis for V : $\left\{ \underset{E_{11}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{E_{12}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{E_{21}}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{E_{22}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\} = \beta$

$$T(E_{11}) = E_{11} = 1E_{11} + 0E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{12}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = E_{21} = 0E_{11} + 0E_{12} + 1E_{21} + 0E_{22}$$

$$T(E_{21}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{12} = 0E_{11} + 1E_{12} + 0E_{21} + 0E_{22}$$

$$T(E_{22}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = E_{22} = 0E_{11} + 0E_{12} + 0E_{21} + 1E_{22}$$

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ is self adjoint, studied in Session 32

$\lambda = \underbrace{1, 1, 1,} \; -1$

$v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Thus T is also self-adjoint.

Eigenvectors of T: Translate to matrices

$v \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ similarly, for ~~many~~ other vectors.

4 (a) $V = \mathbb{R}^3$ $\quad W = \{(x, y, z) : x + 3y - 2z = 0\}$

$\langle \cdot, \cdot \rangle$: dot-product

Need : <u>ONB</u> of W.

First find a basis of W : 2 free vars.

x, y free $\quad x = 1, y = 0$ $\quad x = 0, y = 1$

$\left\{ \begin{array}{l} (1, 0, 1/2) \\ (0, 1, 3/2) \end{array} \right\}$ is a basis of W.

Use G-S to change to an ONB : $\{u_1, u_2\}$

<u>Orthogonal projection</u> of u $(2,1,3)$ onto W

$$u_p = \langle u, u_1 \rangle u_1 + \langle u, u_2 \rangle u_2$$

Orthogonal proj. thm. $\quad u_p$ is closest to W among all elements in W.