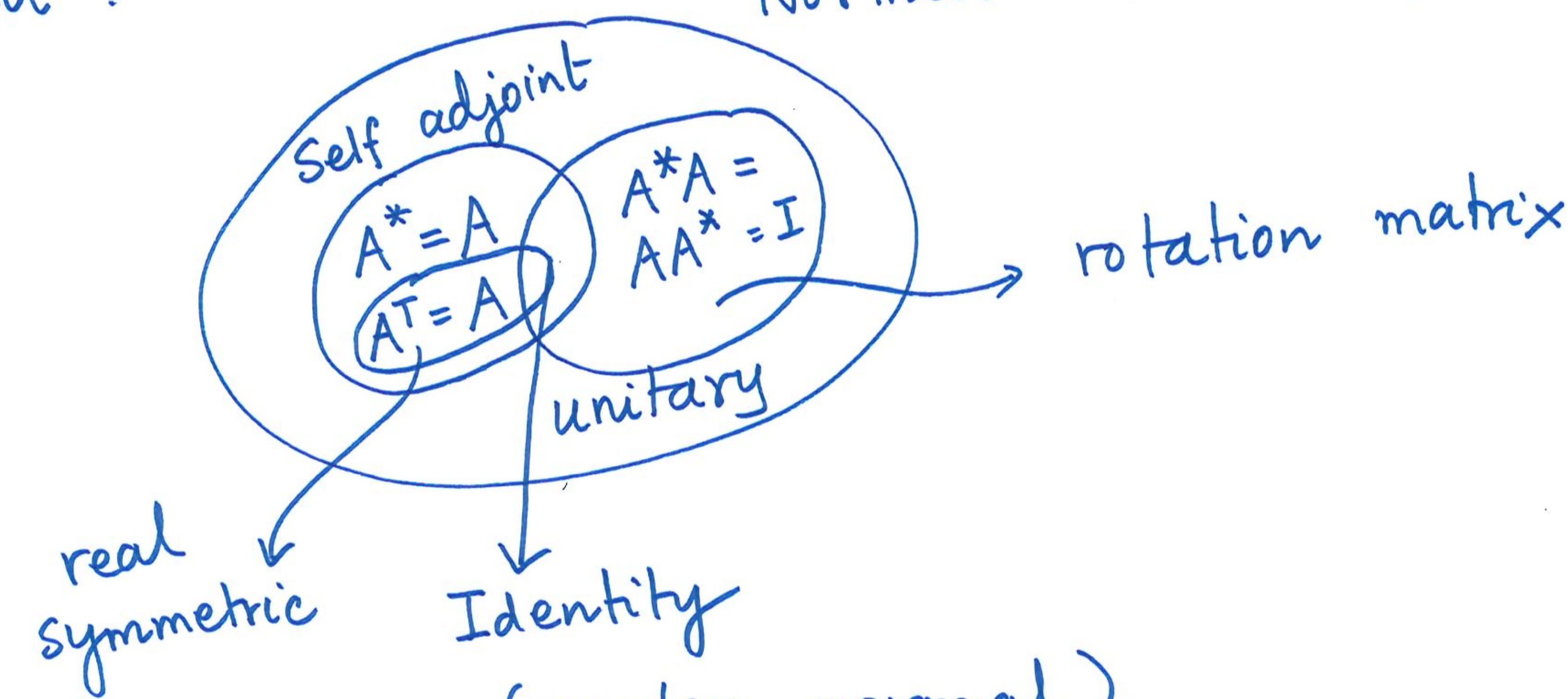
University of Idaho

MATH 430
Advanced Linear Algebra

Session 34

Recall : Normal $A^*A = AA^*$



(complex normal)

Theorem : Every ~~real symmetric~~ self adjoint matrix is **diagonalizable** via an orthogonal (unitary) matrix.

Spectral Theorem

In other words, every real symmetric matrix is orthogonally equivalent to a diagonal matrix.

A - real symmetric $\rightarrow$ $\mathcal{U}^* A \mathcal{U} = D$

$$\boxed{A = \mathcal{U} D \mathcal{U}^*}$$

$\mathcal{U}$ - unitary (orthogonal)
columns of $\mathcal{U}$: e-vectors of A

D - diagonal (entries are e-values)

Conic Sections (6.5) : An application of the Spectral Theorem.

$ax^2 + bxy + cy^2 = M$

Example : Consider $f(x,y) = 2x^2 - 4xy + 5y^2 = 36$

Goal : Change variables $(x,y) \longrightarrow (x', y')$ such that $-4xy$ (cross-term) disappears.

$2x^2 - 4xy + 5y^2$ is an example of a quadratic form (every term is of degree 2).

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{array}{cc} x^2 & xy \\ \end{array}$ $\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ $\begin{array}{cc} yx & y^2 \end{array}$. A is symmetric. $A^T = A$.

$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ symmetric λ_1, λ_2 : e-values of A

$X^T A X$

$= \underset{1\times2}{\begin{bmatrix} x & y \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \cdots = 2x^2 - 4xy + 5y^2$

E-values of A : $\begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0 \cdots \Rightarrow \lambda = 1, 6$

E-vectors of A : $\lambda = 1$: $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$

$y_1 - 2y_2 = 0$ $\quad v_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓ $\quad \|v_1\| = \sqrt{5}$

$\lambda=6$: $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ row 2 $\Rightarrow$ $-2y_1 - y_2 = 0$

$v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\|v_2\| = \sqrt{5}$

$\langle v_1, v_2 \rangle = -2 + 2 = 0$

The ON set of e-vectors is $\left\{ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \right\}$

Let $Q = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. Q is orthogonal.

$$Q^T A Q = D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

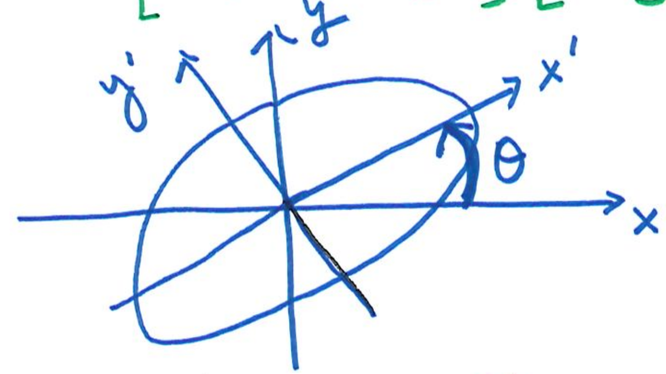
Define a set of new variables by

$$X' = \begin{bmatrix} x' \\ y' \end{bmatrix} := Q^T X = \overset{Q^T}{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X = QX'$$

$$x' = 2/\sqrt{5}\, x + \frac{1}{\sqrt{5}} y$$

$$y' = -1/\sqrt{5}\, x + 2/\sqrt{5}\, y$$



$$2x^2 - 4xy + 5y^2 = X^T A X = (QX')^T A (QX')$$

$$= X'^T \underline{Q^T A Q} X' = X'^T D X'$$

$$= [x' \;\; y'] \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \boxed{1 \cdot x'^2 + 6y'^2 = 36}$$

Ellipse.

$$\lambda_1 x'^2 + \lambda_2 y'^2 = M$$

$$Q = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\theta = \cos^{-1}\left(2/\sqrt{5}\right)$$

$$X = Q X'$$

Sec. 6.5 The above is called the principal axis theorem.