University of Idaho

MATH 430
Advanced Linear Algebra

Session 35

Unitary/Orthogonal Operators

Definition Let $T : V \longrightarrow V$ (linear)
If $\|T(x)\| = \|x\|$ for all $x \in V$, then T is

(i) unitary if $\mathbb{F} = \mathbb{C}$
(ii) orthogonal if $\mathbb{F} = \mathbb{R}$.

By definition, unitary operators are norm preserving

In $\mathbb{R}^2$, a rotation or a reflection preserves norm and are therefore, both unitary/orthogonal.

Properties of unitary operators: $T: V \longrightarrow V$

<u>Theorem</u> (i) T is unitary $\Longleftrightarrow$ $T \circ T^* = T^* \circ T = I$

(ii) T is unitary $\Longleftrightarrow$ $\langle T(x), T(y) \rangle = \langle x, y \rangle$.

<u>Proof</u>: (i) By definition: $\| T(x) \|^2 = \| x \|^2$

$x = T^* T(x) \ \forall x$

$$\|x\|^2 = \langle I x, x \rangle = \| T(x) \|^2 = \langle T(x), T(x) \rangle = \langle T^* T(x), x \rangle$$

This holds for all x if and only if

$T^* \circ T = I$. Also $T \circ T^* = I$.

(T & T^* are inverses of each other)

(ii) T is unitary $\Longleftrightarrow$ $T \circ T^* = T^* \circ T = I$ by (i)

$$\langle T(x), T(y) \rangle = \langle T^* \circ T(x), y \rangle = \langle x, y \rangle$$

<u>Example</u>: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is <u>symmetric</u> ($A^* = A$) <u>and</u> <u>unitary</u> (orthogonal)

$A^*A = AA^* = I$.

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \quad \text{(characteristic eq.)}$$

$$\Rightarrow \lambda = \pm 1$$

$\lambda = 1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an e-vector

$\lambda = -1$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -x_2 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an e-vector

$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$ is an ON set of e-vectors.

- Symmetric (self-adjoint) : ON set of e-vectors
- Eigenvalues have absolute value 1 : unitary.

Theorem: All e-values of an unitary operator have absolute value 1.

Proof: $T : V \longrightarrow V$, T is unitary.
Let x be an e-vector of T for some eigenvalue λ. $x \neq \vec{0}$

Thus, $T(x) = \lambda x \implies \|T(x)\| = |\lambda| \, \|x\|$

Since T is unitary $\|T(x)\| = \|x\|$

Equating both gives $|\lambda| \, \|x\| = \|x\|$

$$\implies |\lambda| = 1 \quad \text{since } x \neq \vec{0}.$$

$\square$

The converse is false.
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\lambda = \underline{1, 1}$ $A^*A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$; A is not unitary.

This leads to

Theorem: $T : V \to V$ is linear and V is a **real** inner product space. Then V has
(1) an ONB of e-vectors of T with corresponding
(2) e-values of absolute value 1 **if and only if** T is **both** self-adjoint and orthogonal.

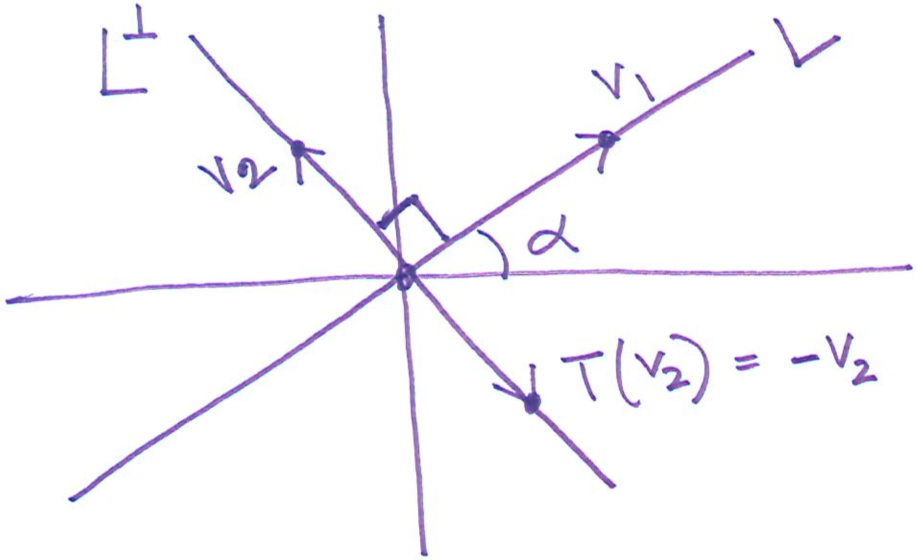
Unitary transformations in $\mathbb{R}^2$:

(a) Rotation: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ <u>rotates</u> counterclockwise by θ. Matrix of T is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A is unitary. $\det(A) = 1$.

(b) Reflection about a line through $(0,0)$.



L: line making an angle α with the x-axis.

$L^\perp$: perpendicular to L

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ reflects about the line L.

v_1 : unit vector on L

v_2 : " " " $L^\perp$

$$\boxed{\begin{aligned} T(v_1) &= v_1 \\ T(v_2) &= -v_2 \end{aligned}}$$

Let $v_1 = (\cos\alpha, \sin\alpha) \in L$ $\qquad \|v_1\| = 1$

$v_2 = (-\sin\alpha, \cos\alpha) \in L^\perp$ $\qquad \|v_2\| = 1$

v_1, v_2 are e-vectors of T for e-values $1, -1$ respectively.

$\underbrace{\{v_1, v_2\}}_{\gamma}$ is an ONB of $\mathbb{R}^2$ of e-vectors of T.

$[T]_\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Let $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the std. basis. Ques: $[T]_\beta$?

$$Q = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

?

$$Q^{-1} [T]_\beta Q = [T]_\gamma \qquad Q^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Q^{-1}: takes back to γ

$[T]_\beta$: matrix w.r.t. std. basis

Q: takes to the std. basis

$$A = [T]_\beta = Q [T]_\gamma Q^{-1} = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

A

reflection matrix about L

$\det(A) = -1$

Theorem: Let T be an orthogonal operator on $\mathbb{R}^2$. Let β be the std. basis of $\mathbb{R}^2$, and $[T]_\beta$ is denoted by A.

Then

(a) T is a rotation & $|A| = 1$

OR

(b) T is a reflection about a line through $(0,0)$ & $|A| = -1$.