MATH 430
Advanced Linear Algebra

Session 36

Singular Value Decomposition

So far: diagonalization of square matrices

- $A \in M_{n \times n}$ has n l. i. e-vectors $\Rightarrow$

A is diagonalizable.

$$Q = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \qquad Q^{-1} A Q = \overset{D}{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}$$

$$A = Q D Q^{-1}$$

- A is self-adjoint or normal then

$$Q^* A Q = \overset{D}{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}} \qquad Q \text{ - unitary}$$

$$A = Q D Q^* \qquad Q^{-1} = Q^*$$

A is similar to D (diagonal)

What if $A \in M_{m\times n}$ where $m \neq n$?

Can we find a <u>decomposition</u> of A into simpler matrices? Desire: use unitary matrices and diagonal matrices.

Let $A_{m\times n}$ have rank $= r$; $r \leq \min(m,n)$

<u>Singular Value Decomposition</u> : A factorization

$$A_{m\times n} = U_{m\times m} \Sigma_{m\times n} V^{*}_{n\times n}$$

where U & V are unitary and Σ

$$\Sigma_{ij} = \begin{cases} \sigma_i & i = j \leq r \\ 0 & \text{otherwise} \end{cases}$$

is called a $\underbrace{\text{singular value decomposition}}_{\text{SVD}}$ of A.

Remark : $U = V \Rightarrow A$ is self adjoint (real field) or A is normal (complex field)

$r < m < n$:

$$\begin{bmatrix} & A & \end{bmatrix}_{m \times n} \qquad \Sigma = \begin{array}{c} \\ \\ \\ m-r \end{array} \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\overleftrightarrow{\quad r \quad}} & (n-r) \\ \sigma_1 & & & & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ & & & \sigma_r & \\ \hline \multicolumn{4}{c}{0} & 0 \quad 0 \quad 0 \end{array}\right]_{m \times n}$$

$r = n < m$

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix}_{m \times n} \qquad \Sigma = \begin{array}{c} \overleftrightarrow{\quad n = r \quad} \\ \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ 0 & & \ddots & \\ & & & \sigma_r \\ 0 & \cdots & & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \updownarrow n \\ \\ \updownarrow m-r \\ \\ \end{array}$$

Theorem 6.26/6.27

The singular value decomposition exists for any matrix $A \in M_{m\times n}$.

Let $A \in M_{m\times n}$. Consider A^*A.

Note that A^*A is self-adjoint

$$(A^*A)^* = A^*(A^*)^* = A^*A$$

Suppose we have the SVD : $A = U\Sigma V^*$

$$A^*A = (U\Sigma V^*)^* U\Sigma V^*$$

$$= (V^*)^* \Sigma^* \underbrace{U^*U}_{=I} \Sigma V^*$$

$U^*U = I$

$$\boxed{A^*A = V\Sigma^*\Sigma V^*}$$

diagonal

$$\underbrace{\underset{n\times m}{\Sigma^*}\ \Sigma_{m\times n}}_{n\times n} = \begin{bmatrix} \sigma_1\bar\sigma_1 & & & & & & \bigcirc \\ & \sigma_2\bar\sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_r\bar\sigma_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ \bigcirc & & & & & & 0 \end{bmatrix}_{n\times n}$$

Assume $\text{rank}(A) = r$

$$= \begin{bmatrix} |\sigma_1|^2 & & & & & & \bigcirc \\ & |\sigma_2|^2 & & & & & \\ & & \ddots & & & & \\ & & & |\sigma_r|^2 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ \bigcirc & & & & & & 0 \end{bmatrix}$$

Diagonal $\leftarrow$

A^*A is diagonalizable by the unitary matrix V.

$$A^*A = V \underbrace{\Sigma^*\Sigma}_{\text{diagonal}} V^* = V D V^*$$

$\Rightarrow \Sigma^*\Sigma$ contains the e-values of A^*A.

Note that the e-values of A^*A are the squares (for real field) of the singular values of A.

What is V made up of ?

The columns of V will be the orthonormal e-vectors of A^*A.

Example Find the SVD of $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}_{2\times 3}$

$\text{rank}(A) = 1$

$$A^* A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

E-values of $A^* A$: $\lambda_1 = 6$, $\lambda_2 = \lambda_3 = 0$

Singular values of A : $\sigma_1 = \sqrt{6}$.

$$\underset{2\times 3}{\Sigma} = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Orthonormal e-vectors of A^*A :

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} & \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} & \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \\ \lambda_1 = 6 & \lambda_2 = 0 & \lambda_3 = 0 \end{array}$$

$$\underset{2\times3}{A} = \underset{2\times2}{U} \underset{2\times3}{\Sigma} \underset{3\times3}{V^*}$$

$$U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$$

$$V = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

$U = ?$ $\quad U$ satisfies

$$A V = U \Sigma V^* V$$

$$A V = U \Sigma$$

$$\Rightarrow A v_i = u_i \sigma_i \qquad \sigma_i \neq 0$$

$$\Rightarrow u_i = \frac{1}{\sigma_i} A v_i$$

$$u_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$U = \begin{array}{cc} u_1 & ? \\ \left[\begin{array}{c} 1/\sqrt{2} \\ 1/\sqrt{2} \end{array}\right. & \left.\begin{array}{c} | \\ u_2 \\ | \end{array}\right] \end{array}$$

Extend $u_1 \rightarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ to an ONB of $\mathbb{R}^2$

Can take $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ Check $\langle u_1, u_2 \rangle = 0$

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Then $U \Sigma V^* = A$