University of Idaho

MATH 430
Advanced Linear Algebra

Session 37

Singular Value Decomposition

Recall : $A \in M_{m\times n}$ $\quad m \neq n$

$$\underset{m\times n}{A} = \underset{m\times m}{U} \underset{m\times n}{\Sigma} \underset{n\times n}{V^*}$$

U, V : unitary

$\Sigma_{m\times n}$: singular values along the diagonal.

$$\Sigma = \begin{array}{c} \uparrow \\ m \\ \downarrow \end{array} \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_r & & & \\ & 0 & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$
$$\longleftarrow n \longrightarrow$$

$r = \text{rank}(A)$

$$U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix}$$

$$V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

Real case:

$$A = u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \cdots + u_r\sigma_r v_r^T$$

σ_i : singular values.

u_i : left singular vectors

v_i = right singular vectors.

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Example $\underset{2\times 3}{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ Find a SVD of A.

$\boxed{A^*A} = V\Sigma^*\Sigma V^T$

$\Sigma^*\Sigma = \begin{bmatrix} \sigma_1^2 & & & & 0 \\ & \sigma_2^2 & & & \\ & & \ddots & & \\ & & & \sigma_r^2 & \\ 0 & & & & 0 \ddots 0 \end{bmatrix}$

Self-adjoint

σ_i^2 : eigenvalues of A^*A

$\sigma_i = ?$

$\underset{2\times 3}{A} = \overset{2\times 2}{U}\underset{2\times 3}{\Sigma} V^*$

$\checkmark U = ?$

$\checkmark \Sigma = ?$

$\checkmark \underset{3\times 3}{V} = ?$

$A^TA = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Self adj.

e-values of A^TA : $\lambda = \underbrace{3, 1}_{\sigma_i^2}, 0$

Singular values of A are

$\sigma_1 = \sqrt{3}, \sigma_2 = 1$

$\underset{2\times 3}{\Sigma} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Next, find V. V is 3×3

$\langle v_i, v_j \rangle = 0$ $i \neq j$

$\lambda = 3$: $(A^T A - 3I)\vec{x} = 0$, $\vec{x} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \longrightarrow v_1$

$\lambda = 1$: $(A^T A - I)\vec{x} = 0$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \longrightarrow \vec{x} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \longrightarrow v_2$

$\lambda = 0$: $A^T A x = 0$ $\vec{x} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \longrightarrow v_3$

$$V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$ is unitary. (orthogonal)

$$\underset{2\times3}{A} = \underset{2\times2}{U}\,\Sigma\, V^* \Rightarrow AV = U\Sigma$$

$$\Rightarrow A v_i = \sigma_i u_i$$

$$\Rightarrow u_i = \frac{1}{\sigma_i} A v_i, \quad \sigma_i \neq 0$$

$$U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}_{2\times2}$$

↑ unitary

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\langle u_1, u_2 \rangle = 0$

$$U = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

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$A = U \Sigma V^*$

Alternative way : Consider $A A^*$

$$A A^* = U \Sigma \underbrace{V^* V}_{= I} \Sigma^* U^* = U \underbrace{\Sigma \Sigma^*}_{\begin{bmatrix} \sigma_1^2 & & & & 0 \\ & \sigma_2^2 & & & \\ & & \ddots & & \\ & & & \sigma_r^2 & \\ 0 & & & & \ddots 0 \end{bmatrix}} U^*$$

Diagonalization of $A A^*$

AA^* is self adjoint.

U is unitary and diagonalizes A.

Orthonormal e-vectors of $A A^*$ form the columns of U.

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4(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}_{4\times 2} = \underset{4\times4}{U} \, \underset{4\times2}{\Sigma} \, \underset{2\times2}{V^*}$

$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Option 1 Diagonalize A^TA, find Σ, V.

$V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$ $U = \begin{bmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & ? & ? \end{bmatrix}$

$u_1 = \frac{1}{\sigma_1} A v_1$, $u_2 = \frac{1}{\sigma_2} A v_2$

$\{u_1, u_2\} \xrightarrow{\text{extend}} \{u_1, u_2, u_3, u_4\}$: an ON set

Option 2 (for find $\mathcal{U}$) :

Find V. Find Σ as in option 1.

Then work with AA^T.

$$AA^T = \mathcal{U} \Sigma \Sigma^* \mathcal{U}^*$$

Calculate e-vectors of AA^T ; these form the columns of $\mathcal{U}$.

Caution : Once you find the columns of $\mathcal{U}$ $\{u_1, \dots, u_4\}$ make sure to pick the e-vectors that satisfy $\boxed{u_i = \frac{1}{\sigma_i} A v_i}$ $i = 1, 2$