University of Idaho

MATH 430
Advanced Linear Algebra

Session 38

Ch: 7 Jordan Canonical Form

$A \in M_{n \times n}$; A is <u>not</u> diagonalizable.

$\Downarrow$

A has a repeated eigenvalue λ s.t.

max. # l. i. e-vectors for λ < algebraic multiplicity of λ

or, $\dim(E_\lambda)$ < "

Recall : Diagonalization ~~alwo~~ allows us to look at A through a simple structure

$$A = P \boxed{D} P^{-1}.$$

SVD : $A \in M_{m \times n}$

$$A = U \Sigma V^*$$

Desire : For a non-diagonalizable square matrix A we wish to find an invertible M and a matrix J (which has a simple structure s.t.

$$M^{-1} A M = J. \qquad A \sim J$$

or $$A = M J M^{-1}$$

Similar matrices : A & B are similar if $\exists$ an invertible P s.t.

$$A = P B P^{-1} \qquad A \sim B$$

<u>Theorem</u> : For any matrix $A_{n\times n}$ with <u>real</u> <u>eigenvalues</u>, $\exists$ exists an invertible M s.t.

$$M^{-1}AM = J = \begin{bmatrix} \boxed{J_1} & \vec{O} & \cdots & \vec{O} \\ \vec{O} & \boxed{J_2} & & \vec{O} \\ \vdots & \vdots & \ddots & \\ \vec{O} & \vec{O} & \vec{O} & \boxed{J_\ell} \end{bmatrix} \rightarrow \text{block diagonal matrix}$$

$\vec{O}$: zero matrix

$A \sim J$

Each J_i is either $[\lambda]_{1\times 1}$ or $\begin{bmatrix} \lambda & 1 & & O \\ & \lambda & 1 & \\ & & \ddots & 1 \\ O & & & \lambda \end{bmatrix} \rightarrow$ square

where λ : e-value of A.

J is called the Jordan canonical form of A.

Each J_i is a Jordan block for λ.

How to find M?

Suppose that A has a single evalue λ.
Suppose that A has a single block J.

$$J = \begin{bmatrix} \lambda & 1 & & \mathbf{0} \\ & \lambda & 1 & \\ & & \ddots & 1 \\ \mathbf{0} & & & \lambda \end{bmatrix} \qquad M^{-1} A M = J \;\Rightarrow\; AM = MJ$$

Let $M = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

$$A \begin{bmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \cdots & v_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & & | \\ v_1 & v_2 & v_3 & \cdots & v_n \\ | & | & & & | \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 & & \mathbf{0} \\ 0 & \lambda & 1 & & \\ 0 & 0 & \lambda & & \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & & & \lambda \end{bmatrix}$$

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$A v_1 = \lambda v_1 \Rightarrow$ v_1 is an e-vector of λ.

$(A - \lambda I) v_1 = 0$ ①

$A v_2 = v_1 + \lambda v_2$

$\Rightarrow (A - \lambda I) v_2 = v_1$

$v_2, v_3, \cdots, v_n$ are called **generalized eigenvectors**.

$A v_3 = v_2 + \lambda v_3$

$\Rightarrow (A - \lambda I) v_3 = v_2$

$\vdots$

$A v_n = v_{n-1} + \lambda v_n$

$\Rightarrow (A - \lambda I) v_n = v_{n-1}$

② $(A - \lambda I)^2 v_2$

$= (A - \lambda I) \underbrace{(A - \lambda I) v_2}_{v_1}$

$= (A - \lambda I) v_1$

$= 0$

③ $(A - \lambda I)^3 v_3 = (A - \lambda I)^2 (A - \lambda I) v_3$

$= (A - \lambda I)^2 v_2 = 0$

Example : $A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & -1 \\ -2 & 2 & -2 \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -2 & 1 \\ 0 & -\lambda & -1 \\ -2 & 2 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda^3 = 0 \Rightarrow \lambda = 0$ is an evalue of multiplicity $\boxed{3}$

E-vectors : $\begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & -1 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 - 2x_2 + x_3 = 0$ $\qquad 2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$

$x_3 = 0$

$-2x_1 + 2x_2 - 2x_3 = 0$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the only l.i. evector for $\lambda = 0$.

Find 2 generalized e-vectors :

Solve $(A - 0I)\,V_2 = V_1$

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & -1 \\ -2 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$2x_1 - 2x_2 + x_3 = 1 \longrightarrow 2x_1 - 2x_2 = 2$

$-x_3 = 1 \Rightarrow x_3 = -1$

$\Rightarrow x_1 - x_2 = 1$

$x_1 = 1$ gives $x_2 = 0$

$$\underbrace{\cancel{-2x_1 + 2x_2 - 2x_3 = 0}}$$

$-x_1 + x_2 - x_3 = 0$

$-x_1 + x_2 = x_3 = -1$

$x_1 - x_2 = 1$

$$V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

generalized e-vector

v_3 is obtained from

$$(A - 0I)v_3 = v_2 \Rightarrow A v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & -1 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$2x_1 - 2x_2 + x_3 = 1$$
$$x_3 = 0$$
$$\cancel{-2x_1 + 2x_2 - 2x_3 = -1}$$

$$2x_1 - 2x_2 = 1$$
$$x_1 - x_2 = 1/2$$

$$v_3 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$$ ← another generalized e-vector.

e-vector

$$\text{Let } M = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1/2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

gen. e-vectors

Then $M^{-1}AM = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = J$ Jordan form with a single Jordan block.