University of Idaho

MATH 430
Advanced Linear Algebra

Session 39

Jordan canonical form

Example: $A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}$ Write A in Jordan form.

E-values: $\begin{vmatrix} 3-\lambda & 1 & -2 \\ -1 & -\lambda & 5 \\ -1 & -1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 3, 2, 2$

$\lambda = 3$: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \Rightarrow V_{\lambda=3} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$V_{\lambda=3}$

$\lambda = 2$:
$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_{\lambda=2} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$$; only one l.i. e-vector.

eq(3) = − eq(1)

~~3 eqs., 2~~

3 unknowns, 2 eqs. ⇒ 1 degree of freedom.

Find a generalized e-vector for $\lambda = 2$:

$$(A - 2I)\, v_g = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{v_g} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$$

$$v_g = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} | & | & | \\ v_{\lambda=3} & v_{\lambda=2} & v_g \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

($\lambda=3$ under the first column; $\lambda=2$ under the second and third columns)

Then $M^{-1} A M = \left[\begin{array}{c|cc} 3 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right] = J$

Jordan form of A $\rightarrow$

Two Jordan blocks.

If $M = \begin{bmatrix} | & | & | \\ v_{\lambda=2} & v_g & v_{\lambda=3} \\ | & | & | \end{bmatrix}$ then $M^{-1} A M = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 3 \end{array}\right]$

Example: Let $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$

Find M s.t. $M^{-1}AM$ is in Jordan form.

Solution: Characteristic polynomial is $(\lambda-2)^3(\lambda-3)$

$\Rightarrow$ E-values of A are $\lambda = 2, 2, 2, 3$

$\lambda = 3$: $\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

R2: $x_3 = 0$, R4: $x_2 = 0$ R3: $x_2 - 2x_3 = 0$

R1: $-x_1 - x_2 + x_4 = 0$

$\Rightarrow x_1 = x_4$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\hookrightarrow$ $V_{\lambda=3}$

$\lambda = 2$:

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

R1 : $x_2 = x_4$

R2 : $x_2 = x_3$

$\Rightarrow x_2 = x_3 = x_4$

x_1 – anything.

R1 = R4, R2 = R3

$\dim E_{\lambda=2} = 2$ $\Rightarrow$ 2 degrees of freedom

$\Rightarrow$ only 2 l.i. e-vectors

regular e-vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

y_1 y_2

$\{y_1, y_2\}$ is l.i.

Remark: Any $y \in \text{span}\{y_1, y_2\}$ is also an e-vector of A for $\lambda = 2$

Need one generalized eigenvector for $\lambda = 2$.

$$(A-2I)\,v_{g_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -x_2 + x_4 &= 1 \\ x_2 - x_3 &= 0 \\ -x_2 + x_4 &= 0 \end{aligned}$$

contradiction, unsolvable.

Take y_2 instead. ... $(A-2I)\,v_{g_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\Rightarrow \quad \begin{aligned} -x_2 + x_4 &= 0 \\ -x_2 + x_4 &= 1 \end{aligned}$$

Same problem — unsolvable.

Want: First & 4th components and 2nd & 3rd components to match.

$$A(\underbrace{c_1 y_1 + c_2 y_2}_{y}) = c_1 A y_1 + c_2 A y_2$$

$$= c_1 \lambda y_1 + c_2 \lambda y_2$$

$$= \lambda (c_1 y_1 + c_2 y_2) = \lambda y$$

$A y = \lambda y$

$\Rightarrow$ y - e-vector

Let us pick: $y = y_1 + y_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$(A - \underbrace{\lambda I}_{2I})v_g = y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Note: $\{y_1, y\}$ or $\{y_2, y\}$ are l.i.

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \overset{y}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$-x_2 + x_4 = 1$$
$$x_2 - x_3 = 1$$
$$x_2 - x_3 = 1$$
$$-x_2 + x_4 = 1$$

$$x_4 = 1 + x_2$$
$$x_3 = x_2 - 1$$

One possible choice is

$x_1 = 0$, $x_2 = 1$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \rightarrow v_g$$

$x_1 = 2$, $x_2 = 1$

Another choice $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$

One possibility for M is

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

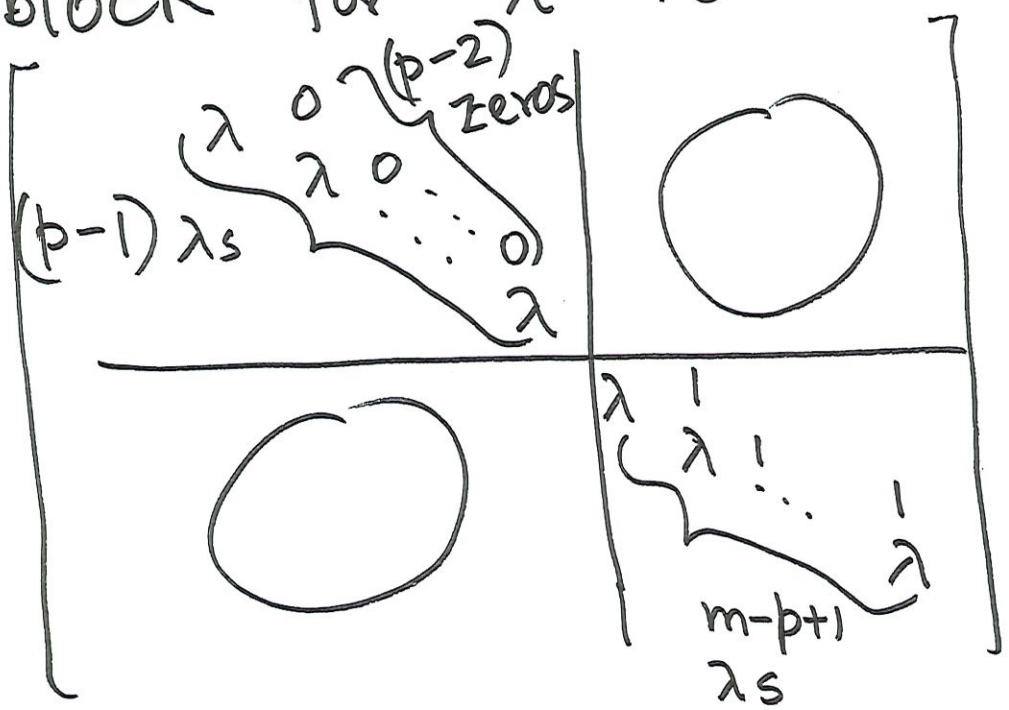
$\lambda = 3$ l.i e-vectors for $\lambda = 2$ y v_g

Then

$$M^{-1} A M = \left[\begin{array}{c|c|cc} 3 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

3 blocks.

Generalizing: Let $A \in M_{n\times n}$ and λ be an e-value of multiplicity m with $p < m$ l.i. e-vectors. Then the Jordan block for λ is



$$(A-\lambda I)v_1 = 0$$
$$\vdots$$
$$(A-\lambda I)v_p = 0$$

} regular e-vectors

$$(A-\lambda I)v_{p+1} = v_p$$
$$\vdots$$
$$(A-\lambda I)v_m = v_{m-1}$$

} start the Jordan chain