MATH 430
Advanced Linear Algebra

Session 4

Linear combination (1.4)

<u>Def</u>: V - vector space over a field $\mathbb{F}$

$S \subseteq V$ (subset)

$x \in V$ is said to be a <u>linear</u> <u>combination</u> of elements in S if $\exists$ scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$

s.t.

$$x = a_1 \odot v_1 \oplus a_2 v_2 + \cdots + a_n v_n$$

where $v_1, v_2, \ldots, v_n \in S$

x need not be in S

<u>Example</u>

$V = \mathbb{R}^3$ $\qquad S = \left\{ \underset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \right\}$

$\mathbb{F} = \mathbb{R}$

$$\underset{x}{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

x is a linear combination of $\underbrace{e_1 \text{ and } e_2}_{\text{vectors in } S}$.

Example: $V = \mathbb{R}^3$, $S = \left\{ \underset{V_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \underset{V_2}{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}} \right\}$. Let $X = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Is X a linear combination of V_1 & V_2? NO

unknown

Rephrase: Can we find a_1, a_2 s.t.

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad ?$$

$$\left.\begin{aligned} a_1 + 2a_2 &= 0 \\ a_1 &= 0 \\ 0 &= 1 \end{aligned}\right]$$

A linear system in two unknowns a_1, a_2.

This has NO solution

NOT POSSIBLE

There is a connection between linear combinations and linear systems.

Def : The set of all linear combinations of vectors in S is called the span of S, denoted by span(S).

S spans $\mathbb{R}^2$. but S does not span $V = \mathbb{R}^3$.

Example $V = \mathbb{R}^3$, $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

span(S)

Consider $\left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$

$$= \underbrace{\left\{ \begin{pmatrix} a \\ b \\ \boxed{0} \end{pmatrix} : a, b \in \mathbb{R} \right\}}_{\text{Span}(S)} = \mathbb{R}^2$$

$\text{Span}(S) = \mathbb{R}^2$; $\text{span}(S) \neq \mathbb{R}^3$

$(0, 0, c)$, $c \neq 0$ is not in span(S)

Theorem (1.5). Let $S \subseteq V$. Then span(S) is a subspace of V.

In the previous example, therefore, $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

Span of the empty set ϕ?

By definition, span$(\phi) = \{\vec{0}\}$.

If span$(S) = V$ then we say that S spans or generates V. Also, S is a spanning or generating set of V. If $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ then S spans $\mathbb{R}^3$.

P_n : set of all polynomials of degree $\leq n$

$P_n(\mathbb{R})$: " " " " " with real coefficients.

$P(\mathbb{R})$: set of all polynomials

Example : $V = P_2(\mathbb{R})$

Is $\text{span}(S) = P_2(\mathbb{R})$?

Show that $S = \{x^2 + 3x - 2,\ 2x^2 + 5x - 3,\ -x^2 - 4x + 4\}$ generates/spans $P_2(\mathbb{R})$.

Any polynomial in P_2 is given by ax^2+bx+c.

where a, b, c are known.

Find α, β, γ s.t. unknowns

$$\alpha(x^2+3x-2)+\beta(2x^2+5x-3)+\gamma(-x^2-4x+4) = ax^2+bx+c$$

Equate x^2: $\alpha + 2\beta - \gamma = a$

x: $3\alpha + 5\beta - 4\gamma = b$

Constant : x^0 : $-2\alpha - 3\beta + 4\gamma = c$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -4 \\ -2 & -3 & 4 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \xRightarrow{\text{check}} \begin{aligned} \alpha &= -8a+5b+3c \\ \beta &= 4a-2b-c \\ \gamma &= -a+b+c \end{aligned}$$