MATH 430
Advanced Linear Algebra

Session 40

Example ①

$$A = \begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{bmatrix}$$

Find a Jordan form for A.

The characteristic polynomial is $(\lambda-2)^2(\lambda-4)^2$.

Eigenvalues are $\lambda = 2, 2, 4, 4$.

$\lambda = 2$:

$$\begin{bmatrix} 0 & -4 & 2 & 2 \\ -2 & -2 & 1 & 3 \\ -2 & -2 & 1 & 3 \\ -2 & -6 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_2 + x_3 + x_4 = 0$

$-2x_1 - 2x_2 + x_3 + 3x_4 = 0$

$-2x_1 - 6x_2 + 3x_3 + 5x_4 = 0$

$2x_1 - 2x_4 = 0 \Rightarrow x_1 = x_4$

$-2x_1 + 6x_2 + 3x_3 + 5x_4 = 0$

$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ are two l.i. evectors for $\lambda = 2$.

$\lambda = 4$: $\begin{bmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Only one l.i. e-vector can be obtained;

$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$

Need another l.i. evector : find a generalized e-vector

Solve $(A-4I)\,v_g = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$v_g = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$ is a generalized e-vector for $\lambda = 4$

can swap

$$\text{Let } M = \left[\begin{array}{cccc} 2 & 0 & \overbrace{0 \quad 1}^{\lambda=4} \\ 1 & 1 & 1 \quad 1 \\ 0 & 2 & 1 \quad 1 \\ \underbrace{2 \quad 0}_{\lambda=2} & & 1 \quad \underset{\downarrow\, v_g}{2} \end{array}\right] \qquad J = \left[\begin{array}{c|c|cc} 2 & 0 & 0 & 0 \\ \hline 0 & \boxed{2} & 0 & 0 \\ \hline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right] \to \text{Jordan form}$$

3 blocks

$\hookrightarrow M^{-1}AM$

Another choice :

$$M=\begin{bmatrix}2 & \overset{\vec{v}}{0} & \overset{\vec{v}_g}{1} & 0\\ 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 2\\ 2 & 1 & 2 & 0\end{bmatrix}$$

$\lambda=2$ e-vector $\qquad \lambda=4 \qquad \lambda=2$ e-vector

Then $M^{-1}AM=J=\left[\begin{array}{c|cc|c}2&0&0&0\\ \hline 0&4&1&0\\ 0&0&4&0\\ \hline 0&0&0&2\end{array}\right]$

Another choice $\quad$ can swap

$$M=\begin{bmatrix}0 & 1 & 2 & 0\\ 1 & 1 & 1 & 1\\ 1 & 1 & 0 & 2\\ 1 & 2 & 2 & 0\end{bmatrix}$$

$\lambda=4 \qquad \lambda=2$

Then $M^{-1}AM=J=\left[\begin{array}{cc|c|c}4&1&0&0\\ 0&4&0&0\\ \hline 0&0&2&0\\ \hline 0&0&0&2\end{array}\right]$

HW 10

#4 $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ Find a Jordan form for A.

$\lambda = \underbrace{1, 1, 1,}\ 2$

$\lambda = 2$: $V_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1$: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $V_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

R2: $x_1 = 0$, R3: $x_2 = 0$, R4: $x_4 = 0$

x_3 - anything.

Cannot find any more eigenvectors that are l.i. with V_1.

Need 2 generalized e-vectors for $\boxed{\lambda = 1}$:

Solve $(A - I)V_{g_1} = V_1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

R2 $\Rightarrow$

$x_1 = 0$, R3: $2x_2 = 1 \Rightarrow x_2 = 1/2$, R4: $x_4 = 0$

x_3 - anything Take $V_{g_1} = \boxed{\begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}}$

Solve $(A - I)V_{g_2} = V_{g_1}$

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$

R2: $x_1 = 1/2$

R3: $x_2 = 0$

R4: $x_4 = 0$

$V_{g_2} = \boxed{\begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$

$$M = \begin{bmatrix} 0 & \overbrace{0 \quad 0 \quad \tfrac{1}{2}}^{\text{cannot change the order}} \\ 0 & 0 \quad \tfrac{1}{2} \quad 0 \\ 0 & 1 \quad 0 \quad 0 \\ \underbrace{1}_{\lambda = 2} & \underbrace{0}_{\substack{\lambda=1 \\ \text{e-vector}}} \quad \underbrace{0 \quad 0}_{\substack{\text{gen. evectors} \\ \text{for } \lambda = 1}} \end{bmatrix}$$

Then

$$M^{-1} A M = \underset{\substack{\uparrow \\ \text{Jordan form}}}{J} = \left[\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hw 9.

4(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}_{4\times 2}$ Find the SVD.

$$A = \underset{4\times4}{U} \; \underset{4\times2}{\Sigma} \; V^T_{2\times2}$$

Find e-values of $\underbrace{A^T A}_{V \Sigma^2 V^T}$: 5, 1

Singular values : $\sigma_1 = \sqrt{5}$, $\sigma_2 = 1$

$$\Sigma = \left[\begin{array}{cc} \sqrt{5} & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \qquad V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$$

v_1, v_2 : ON e-vectors of $A^T A$

$$\underset{4\times 4}{U} = \begin{bmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{bmatrix}$$

$\cancel{\sigma} = \sqrt{5}, 1$

$$u_1 = \frac{1}{\sqrt{5}} A v_1$$

$$u_2 = \frac{1}{1} A v_2 = A v_2$$

Consider $\underbrace{\boxed{A A^T}}_{4\times 4} \rightarrow$ symmetric $= U \Sigma^2 U^T$

E-values of AA^T : $\underbrace{5, 1}_{u_1, u_2}, \underbrace{0, 0}$ $\rightarrow$

u_3, u_4 : the ON e-vectors for $\lambda = 0$