University of Idaho

MATH 430
Advanced Linear Algebra

Session 5

Linear dependence & independence (1.5) ①

Example $V = \mathbb{R}^3$ $\quad S = \left\{ \underset{u_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{u_3}{\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}} \right\}$

nontrivial rep. of zero

$u_3 = 2u_1 + u_2$ $\quad$ or, $\quad \underbrace{2u_1 + u_2 - u_3}_{\text{linear combination with nonzero scalars}} = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\{u_1, u_2, u_3\}$ is a linearly dependent set

$a_1 = 2, \quad a_2 = 1, \quad a_3 = -1$

l.d.

$S \subseteq V$ is linearly dependent if $\exists\, u_1, u_2, \dots, u_n$ in S and $a_1, a_2, \dots, a_n \in \mathbb{F}$, <u>not all zero</u> s.t.

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = \vec{0}$$

l.i.

Otherwise, S is linearly independent.

The above can be restated as :

A set $S = \{u_1, u_2, \dots, u_n\} \subseteq V$ is linearly independent if

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = \vec{0}$$

implies

$$a_1 = a_2 = \cdots = a_n = 0$$

Trivial representation of $\vec{0}$

Example $V = M_{2\times 2}(\mathbb{R})$ $\quad S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right\}$

Is S l.i. ?

$$\alpha \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{\alpha} & \alpha + \beta \\ \underline{\underline{0}} & \underline{\underline{2\beta}} \end{bmatrix} = \begin{bmatrix} \boxed{0} & 0 \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix}$$

$$\left.\begin{aligned} \alpha &= 0 \\ \alpha + \beta &= 0 \\ 2\beta &= 0 \end{aligned}\right]$$

$\Rightarrow \beta = 0$

Linear system with two unknowns: α, β

$\alpha = \beta = 0$ is the only solution.

$\Rightarrow S$ is l.i.

Example : $V = \mathbb{R}^3$ $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \right\}$

Is S l.d. ?

We already know that S is l.d.

The following is the general method to solve

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \alpha + \beta + 3\gamma &= 0 \\ \alpha + \quad + 2\gamma &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= -2\gamma \\ \beta &= -\gamma \end{aligned} \qquad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -2t \\ -t \\ t \end{pmatrix} \quad t \in \mathbb{R}$$

$\cancel{\alpha + \beta + 3\gamma = 0}$

nontrivial solutions exist

$\Rightarrow$ S is l.d.

Infinitely many solutions

Is $\{\vec{0}\}$ l.d. ?

Yes.

$$a\vec{0} = \vec{0}$$

for some nonzero scalar a.

Any set containing the zero vector is l.d.

$\{\vec{0}, \vec{u}, \vec{v}, \dots\}$

$$\underline{\underline{a}}\,\vec{0} + 0\cdot\vec{u} + 0\cdot\vec{v} + \dots = \vec{0}$$

If ϕ l.d ?

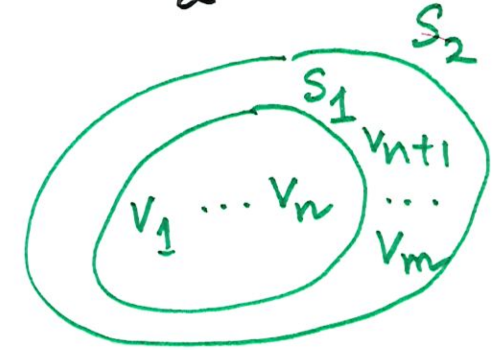
ϕ is l.i

ϕ is NOT l.d. because it is empty ; a l.d. set cannot be empty.

Thus ϕ is l.i.

A set consisting of a single nonzero vector is l.i.

Theorem: Let $S_2 \subseteq V$, and let $S_1 \subseteq S_2$. If S_1 is linearly dependent, then S_2 is also linearly dependent.



COR: Let $S_1 \subseteq S_2 \subseteq V$. If S_2 is l.i., then S_1 is also l.i.

Proof: Since S_1 is l.d., there are vectors $v_1, v_2, \ldots, v_n$ in S_1 and scalars $a_1, a_2, \ldots, a_n$, not all zero, s.t.

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \vec{0} \quad (i)$$

Since $S_1 \subseteq S_2$, $v_1, \cdots, v_n \in S_2$ as well.

Consider $\{v_1, \cdots, v_n, v_{n+1}, v_{n+2}, \cdots, v_m\}$, $\underline{\underline{m \geq n}}$

$\underline{\text{in } S_2}$ where $v_{n+1}, \cdots, v_m \in S_2$.

Consider the linear combination

a's are same as above

$$\underbrace{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n}_{= \vec{0} \text{ by (i)}} + \underbrace{0 \cdot v_{n+1} + 0 \cdot v_{n+2} + \cdots + 0 v_m}_{= \vec{0}}$$

$= \vec{0}$

where not all scalars are zero.

Thus S_2 is also l. d.

$\square$