MATH 430
Advanced Linear Algebra

Session 6

Bases (1.6)

Previously :

$\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$

Span of a set (the <u>spanning set of a vector space</u>)

linear dependence

linear independence

<u>Goal</u> : Find ~~the~~ an "optimal" set that spans a vector space

Today

(1.6) Basis

Example : $V = \mathbb{R}^3$

$S_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ has more elements than needed to span $\mathbb{R}^3$

l.d., spans $\mathbb{R}^3$

$\text{span}(S_3) = \text{span}(S_2) = \mathbb{R}^3$

$S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ "optimal" basis for $\mathbb{R}^3$

l.i., spans $\mathbb{R}^3$

$S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ too few elements to span $\mathbb{R}^3$

l.i., DOES NOT span $\mathbb{R}^3$

Basis : A set B is a <u>basis</u> for a vector space V if

(a) B is linearly independent

(b) B is a spanning set of V (B spans V) $\text{span}(B) = V$

A basis is a linearly independent spanning set.

We saw that in $\mathbb{R}^3$

$$\underbrace{\left\{ \underset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}}_{\text{Standard basis of } \mathbb{R}^3}$$

is a basis.

1. $\mathbb{R}^n$: $\left\{ \overset{e_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \overset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \ldots, \overset{e_k}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \ldots, \overset{e_n}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}} \right\}$

kth position

is the standard basis for $\mathbb{R}^n$

e_k : kth coordinate is 1, every other entry is zero

2. $M_{2\times 2}$: $\left\{ \overset{E_{11}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \overset{E_{12}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{E_{21}}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \overset{E_{22}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a basis of $M_{2\times 2}$

$M_{m\times n}$: E_{ij} is the matrix in $M_{m\times n}$ s.t. the (i,j)th entry is 1 and the rest 0

$\{E_{ij} : 1 \le i \le m, 1 \le j \le n\}$ is a basis of $M_{m\times n}$

Standard basis

3 $P_n(\mathbb{R})$: polynomials with degree $\leq n$, real coefficients

$\{1, x, x^2, \ldots, x^n\}$ is a basis ; called the standard basis.

- Let S be a l.d. subset of V.
- Let $v \in S$ s.t. v is a linear combination of other vectors in S.
- Let $R = S \setminus \{v\}$: Remove v
- $\text{Span}(R) = \text{span}(S)$
- If $S \subseteq V$ and $\text{span}(S) = V$, then S may be l.d.
- A l.d. spanning set can be reduced to a basis

l.d. size(S) = 4

$$S = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\} \xrightarrow{\text{reduce}}$$

l.i. size(B) = 3

$$B = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

- A l.i. set that does not span V can be expanded to a basis

$$L = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\} \xrightarrow{\text{expand}} B = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

l.i.

$\text{Span}(L) \neq \mathbb{R}^3$

$\text{Size}(L) = 2$

l.i.

$\text{Span}(B) = \mathbb{R}^3$

$\text{Size}(B) = 3$

<u>Replacement Theorem</u> : Let V be a vector space. Suppose $S \subseteq V$ and S is a spanning set with $\text{size}(S) = n$.
Suppose that $L \subseteq V$, and L is l.i. with $\text{size}(L) = m$.

Then

$$m \leq n.$$

size of a basis is the <u>dimension</u> of V.

l.i

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is a basis of $\mathbb{R}^3$

Take $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3$.

$$\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \boxed{1\cdot} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \boxed{1\cdot} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \boxed{0\cdot} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

unique rep. of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in terms of vectors in B.