MATH 430
Advanced Linear Algebra

Session 7

Unique representation using a basis

<u>Theorem</u>: $B = \{u_1, u_2, \ldots, u_n\} \subseteq V$ is a basis of V $\Longleftrightarrow$ (if & only if) every vector in V can be expressed <u>uniquely</u> in terms of elements in B.

<u>Proof</u> $(\Rightarrow)$ Let B be a basis of V. Thus <u>B spans</u> V. Let $v \in V$. $\exists\, a_1, \ldots, a_n$ s.t.

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

Suppose there exist <u>different</u> scalars $b_1, b_2, \ldots, b_n$ s.t.

$$v = b_1 u_1 + b_2 u_2 + \cdots + b_n u_n$$

Subtracting

$$\vec{0} = v - v = (a_1 - b_1) u_1 + (a_2 - b_2) u_2 + \cdots + (a_n - b_n) u_n$$

Since B is l. i.

$$\left.\begin{aligned} a_1 - b_1 &= 0 \\ a_2 - b_2 &= 0 \\ &\vdots \\ a_n - b_n &= 0 \end{aligned}\right\} \begin{aligned} a_1 &= b_1 \\ a_2 &= b_2 \\ &\vdots \\ a_n &= b_n \end{aligned}$$

$\Rightarrow$ The representation of v ~~V~~ is unique.

($\Leftarrow$:) Assume that every $v \in V$ can be expressed uniquely in terms of $\overbrace{u_1, \cdots, u_n}^{B}$.

This implies that B spans V.

Let

$$b_1 u_1 + b_2 u_2 + \cdots + b_n u_n = \vec{0} \quad \text{(i)}$$

Since $\vec{0} \in V$, $\vec{0}$ has a unique representation by assumption. We have

$$0 u_1 + 0 u_2 + \cdots + 0 u_n = \vec{0} \quad \text{(ii)}$$

Due to uniqueness, $b_1 = \cdots = b_n = 0 \Rightarrow$ B is l. i

Thus B is a basis

<u>Theorem</u>: (i) Every basis of a vector space has the same number of vectors.

(ii) If the dimension of a vector space is d then every l.i. set of d vectors is a basis.

<u>Dimension</u> : The number of elements in a basis is called the <u>dimension</u> of V. Denoted by

$$\dim(V)$$

<u>Example</u> :

1. $\mathbb{R}^n$, $\dim(\mathbb{R}^n) = n$, basis $= \{e_1, e_2, \dots, e_n\}$
2. $M_{m\times n}(\mathbb{R})$, $\dim(M_{m\times n}) = mn$, basis $= \{E_{ij} ;\ 1 \le i \le m,\ 1 \le j \le n\}$
3. $P_n(\mathbb{F})$, $\dim(P_n(\mathbb{F})) = n+1$, basis $= \{1, x, x^2, \dots, x^n\}$
4. $P(\mathbb{F})$: set of <u>all</u> polynomials

   Basis $= \{1, x, x^2, x^3, \dots\}$ $\quad \dim(P(\mathbb{F})) = \infty$

If a basis has finitely many elements, then the vector space is finite dimensional otherwise it is infinite dimensional.

V - vector space, B - basis for V - ① B is l.i
② B spans V
span$(B) = V$

$\dim(V) = d \Rightarrow$ size$(B) = d$

- Let $G \subseteq V$ s.t. span$(G) = V$, i.e., G is a spanning set. G may be l.d. Let $\underline{\text{size}(G) = n}$.

Then $\boxed{n \geq d}$

If $n = d$ then G is a basis.

- Let $L \subseteq V$ s.t. L is l.i. Let size$(L) = m$. Then $\boxed{m \leq d}$

If $m = d$ then span$(L) = V$ & L is a basis.

spanning sets $\xrightarrow{\text{reduce}}$ basis

linearly independent sets $\xrightarrow{\text{expand}}$ basis

- A basis of a vector space is not unique but each ~~ea~~ basis has exactly d vectors if $\dim(V) = d$.

Find two different bases for $M_{2\times 2}$

1. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
2. ?

Secs. 1.4 – 1.7