University of Idaho

MATH 430
Advanced Linear Algebra

Session 8

Dimension of subspaces

Theorem: Let V be a vector space (finite dimensional) and let W be a subspace of V. Then

$$\dim(W) \leq \dim(V).$$

Equality holds if and only if $W = V$.

Example 1 $V = \mathbb{R}^4$; $W = \{(x_1, x_2, x_3, x_4) : x_1 = x_4, \; x_2 = 0\}$

- W is a subspace
- Basis = ? $x_2 = 0$, x_3 - free, $x_1 = x_4 \Rightarrow x_1$ OR x_4 is free

2 free variables, say x_1 & x_3

$x_1 = 1$, $x_3 = 0$: $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $x_1 = 0$, $x_3 = 1$

$\{\vec{u}_1, \vec{u}_2\}$ is a basis of W.

$\dim(W) = 2$.

Ex. 2 $\quad V = M_{n \times n}$

W = all <u>diagonal</u> matrices in V. $\quad \dim(W) = ?$

hw. show that W is a subspace

$\dim(V) = n^2 \qquad \dim(W) \le n^2$

Basis of W is $\{E_{ii},\ 1 \le i \le n\}$

$$\left\{ \underbrace{\begin{bmatrix} 1 & & O \\ & 0 & \\ O & & \ddots \; 0 \end{bmatrix}}_{E_{11}},\ \underbrace{\begin{bmatrix} 0 & & O \\ & 1 & \\ O & & 0 \ddots 0 \end{bmatrix}}_{E_{22}},\ \cdots,\ \underbrace{\begin{bmatrix} 0 & & O \\ & 0 & \\ O & & \ddots \; 1 \end{bmatrix}}_{E_{nn}} \right\}$$

$\dim(W) = n.$

Ex: 3 $V = \mathbb{R}^5$; $W = \{(a_1, a_2, a_3, a_4, a_5) : a_1 + a_3 + a_5 = 0,\ a_2 = a_4\}$

- W is a subspace of V.
- $\dim(V) = 5 \Rightarrow \dim(W) \leq 5$.

$a_1 = -a_3 - a_5$

$a_1 + a_3 + a_5 = 0 \Longrightarrow a_5 = -a_1 - a_3$ (2 free)

$a_2 = a_4$ (one free)

$u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ $\quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

u_2: $a_1 = 1$, $a_3 = 0$

u_3: $a_1 = 0$, $a_3 = 1$

$\{u_1, u_2, u_3\}$ is a basis, $\dim(W) = \boxed{3}$

Geometric interpretation of subspaces of $\mathbb{R}^2$.

$V = \mathbb{R}^2$ The possible dimension of any subspace W is 0, 1 or 2.

If $\dim(W) = 0$ then $W = \{\vec{0}\} = \text{span}\{\varphi\}$ = the origin

If $\dim(W) = 1$ then W = all scalar multiples of a nonzero vector in $\mathbb{R}^2$ = a line through the origin ($y = mx$)

If $\dim(W) = 2$ then $W = \mathbb{R}^2$.

Ch: 2 Linear transformations. ⑤

So far : abstract vector spaces

Now : study mappings from vector spaces to vector spaces

$$f : V \longrightarrow W$$

V - domain

W - codomain

Ex :

$$f : \underset{V}{\mathbb{R}^3} \longrightarrow \underset{W}{\mathbb{R}^2}$$

$$\vec{x} \longmapsto A\vec{x}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}_{2\times 3}$$

$$f(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ 2x_2 \end{bmatrix}$$

$\vec{x} = (x_1, x_2, x_3)$

$\longmapsto$

$(x_1 + x_2 + x_3, 2x_2) \in \mathbb{R}^2$

f is linear

Definition : $T : V \longrightarrow W$ is called a <u>linear</u> transformation if

(a) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in V$

(b) $T(c\vec{x}) = cT(\vec{x})$ for all $\vec{x} \in V$, $c \in \mathbb{F}$

<u>Ex</u> : $f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $\underline{f(x) = x^2}$

$$f(x_1 + x_2) = (x_1 + x_2)^2 \neq f(x_1) + f(x_2)$$

$f(x_1) = x_1^2$

$f(x_2) = x_2^2$

f is not linear.