University of Idaho

MATH 430
Advanced Linear Algebra

Session 9

Linear Transformations (2.1)

$T : V \longrightarrow W$ $\qquad$ V- domain, W- codomain

T is a linear transformation if

(a) $T(x+y) = T(x) + T(y) \quad \forall\, x, y \in V$

(b) $T(cx) = cT(x) \quad \forall\, x \in V,\ \forall\, c \in \mathbb{F}$

Let $c = 0$.

$\forall\, x \in V$, $T(0x) = T(\vec{0}_V)$

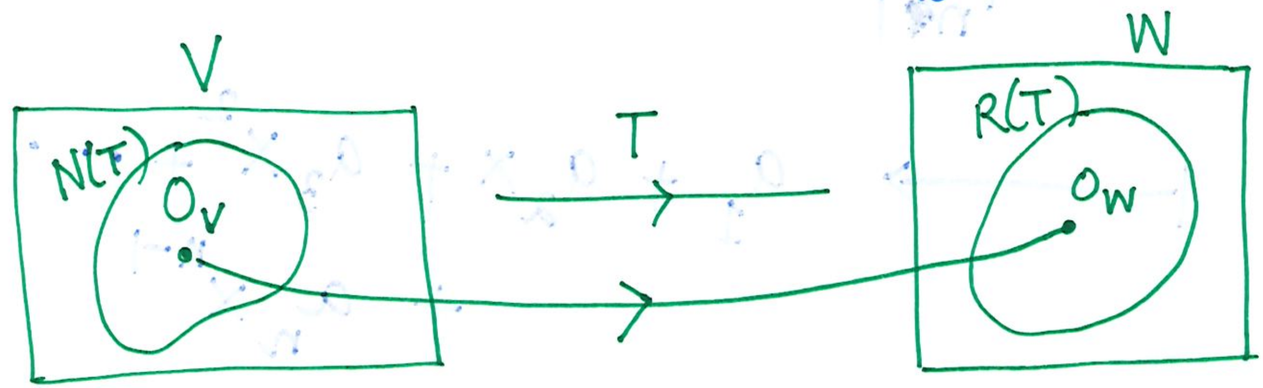
Also, $T(0x) = 0\underbrace{T(x)}_{\in W} = \vec{0}_W$

$\Rightarrow T(\vec{0}_V) = \vec{0}_W$

Zero vector of V is mapped to zero vector of W.

Range of $T = R(T) = \{ y \in W : \exists\, x \in V$ for which $T(x) = y \}$

Null space of $T = N(T) = \{ x \in V : T(x) = \vec{0} \in W \}$



(a) & (b) are together equivalent to

$$T(cx + y) = c\,T(x) + T(y)$$

<u>Example</u> : $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $a = (a_1, a_2, a_3)$

$$T(a_1, a_2, a_3) = (a_1 - a_2 , 2a_3)$$

(a) T is linear.

Let $b = (b_1, b_2, b_3)$

$$T(a+b) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3)$$
$$= (a_1 + b_1 - a_2 - b_2 , 2(a_3 + b_3))$$
$$= (a_1 + b_1 - a_2 - b_2 , 2a_3 + 2b_3)$$

$$T(a) + T(b) = (a_1 - a_2 , 2a_3) + (b_1 - b_2, 2b_3)$$
$$= (a_1 - a_2 + b_1 - b_2 , 2a_3 + 2b_3)$$
$$= T(a+b)$$

$$T(\underline{\underline{ca}}) = T(ca_1, ca_2, ca_3) = (ca_1 - ca_2, 2ca_3)$$

$$cT(a) = c(a_1 - a_2, 2a_3) = (c(a_1 - a_2), c2a_3)$$

$$= (ca_1 - ca_2, 2ca_3)$$

$$\Rightarrow T(ca) = c\,T(a)$$

T is linear.

(b) $N(T) = ?$

$$= \left\{ (a_1, a_2, a_3) : \begin{array}{l} a_1 - a_2 = 0 \\ 2a_3 = 0 \end{array} \right\}$$

$$= \{ (a_1, a_2, a_3) : a_1 = a_2, a_3 = 0 \}$$

$$= \{ (a, a, 0) : a \in \mathbb{R} \}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Theorem 2.1. Let V & W be vector spaces and

$T : V \longrightarrow W$ be linear.

Then $N(T)$ and $R(T)$ are subspaces of V and W, respectively.

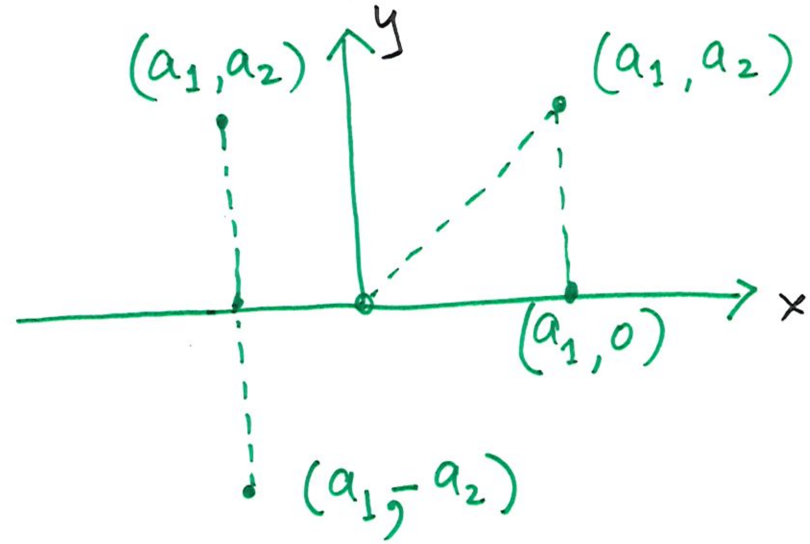
Examples of linear transformations

Geometry

ⓐ Reflection about x-axis :

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(a_1, a_2) = (a_1, -a_2)$

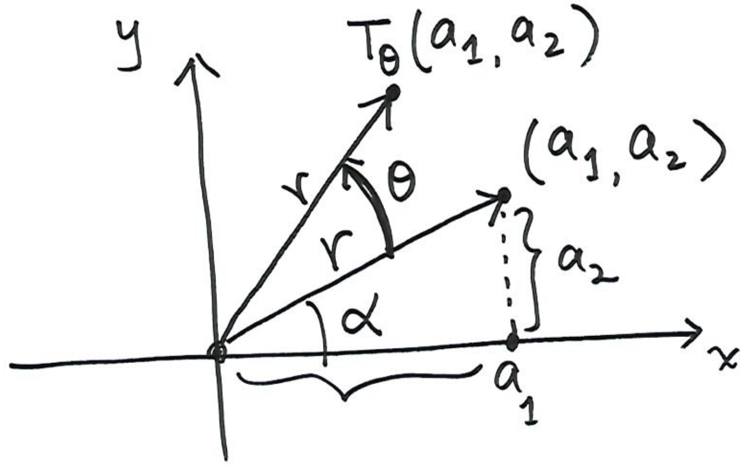


ⓑ $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(a_1, a_2) = (a_1, 0)$

T is the projection on the x-axis

(c) Rotation:



$a_1 = r\cos\alpha$

$a_2 = r\sin\alpha$

$$T_\theta(a_1, a_2) = \left(r\cos(\theta+\alpha),\ r\sin(\theta+\alpha)\right)$$

$$= \left(r\cos\theta\cos\alpha - r\sin\theta\sin\alpha,\ \overbrace{r\cos\alpha}^{a_1}\sin\theta + r\sin\alpha\cos\theta\right)$$

$$= \left(a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta\right)$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation matrix}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

T_θ: rotation by θ counter-clockwise.

In Calculus :

1. $T : V \longrightarrow W$ $\quad$ $V = W = P_n(\mathbb{R})$

$T(f) = f'(t)$

T is linear.

$R(T) = P_{n-1}(\mathbb{R})$

$N(T) = \text{constants} = \text{span}\{1\}$.

2. $V = C(\mathbb{R}) =$ continuous functions

$T : V \longrightarrow \mathbb{R}$ $\quad$ $T(f) = \int_a^b f(t)\,dt \quad \forall f \in V$

then T is linear.