# Practice Problems for Exam I

**MATH 430**

1. (Section 1.2) Let $V = \mathbb{R}^2$ with addition defined by
$$(a_1, a_2) \oplus (b_1, b_2) = (2a_1b_1, a_2 - b_2)$$
and scalar multiplication defined by
$$c \odot (a_1, a_2) = (a_1, ca_2).$$
   (a) Is this a vector space over the field $\mathbb{R}$.
   (b) What is the zero vector in this case?

2. (Section 1.2) Show that the zero vector of a vector space is unique. (See Corollary to Theorem 1.1 in the text).

3. (Section 1.2) Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$, where $F$ is a field. Define addition of elements of $V$ coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$, define
$$c \odot (a_1, a_2) = (a_1, 0).$$
Is $V$ a vector space over $F$ with these operations?

4. (Section 1.2) In $\mathbb{R}^2$ define addition and scalar multiplication as
$(a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$ and
$c \odot (a_1, a_2) = (ca_1, ca_2)$.
Is $\mathbb{R}^2$ a vector space over $\mathbb{R}$ under the above operations? Justify.
(See class notes of Session 2.)

5. (Section 1.2) What is the zero vector of the vector space $M_{3\times 4}$?
(Ans. The $3 \times 4$ zero matrix.)

6. (Section 1.3) $W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$. Is $W$ a subspace of $\mathbb{R}^3$ under componentwise addition and scalar multiplication? Justify.

7. (Section 1.3) Let $A$ be a matrix and denote by $A^t$ the transpose of $A$. A matrix $A$ is said to be symmetric if $A = A^t$. Show that the set of $n \times n$ symmetric matrices form a subspace of $M_{n\times n}(\mathbb{R})$

8. (Section 1.4) Does $x^3 - 3x + 5$ belong to the span of $\{x^3 + 2x^2 - x + 1, x^3 + 3x^2 - 1\}$?
(Yes, $x^3 - 3x + 5 = 3(x^3 + 2x^2 - x + 1) - 2(x^3 + 3x^2 - 1)$.)

9. (Section 1.5) Determine whether the following sets are linearly independent or linearly dependent.
(a) $\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\}$ in $M_{2\times 2}(\mathbb{R})$. (Ans. Linearly dependent.)
(b) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(\mathbb{R})$ (Ans. Linearly independent)

10. (Section 1.6) Do the polynomials $\{x^3-2x^2+1, 4x^2-x+3, 3x-2\}$ generate $P_3(\mathbb{R})$? Justify your answer.

11. (Section 1.6) Is the set $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ a linearly independent subset of $\mathbb{R}^3$? Justify your answer.

12. (Section 1.6) The vectors $u_1=(2,-3,1)$, $u_2=(1,4,-2)$, $u_3=(-8,12,-4)$, $u_4=(1,37,-17)$, and $u_5=(-3,-5,8)$ generate $\mathbb{R}^3$. Find a subset of the set $\{u_1,u_2,u_3,u_4,u_5\}$ that is a basis for $\mathbb{R}^3$.
(Ans. $\{u_1,u_2,u_5\}$)

13. (Section 2.1) For the following, prove that $T$ is a linear transformation, find the bases for $N(T)$ and $R(T)$, and determine whether $T$ is one-to-one or onto.

    (a) $T:\mathbb{R}^2\to\mathbb{R}^3$, $T(a_1,a_2)=(a_1+a_2,0,2a_1-a_2)$.

    (b) $T:M_{2\times3}\to M_{2\times2}$ defined by
    $T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{pmatrix}=\begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0\end{pmatrix}$
    (same as HW 4, #1(b))

14. (Section 2.2) Let $\alpha=\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\}$ and $\beta=\{1,x,x^2\}$.
(a) Define $T:M_{2\times2}\to M_{2\times2}$ by $T(A)=A^{\mathrm{t}}$, where $A^{\mathrm{t}}$ means the transpose of the matrix $A$. Compute the matrix of $T$, i.e., $[T]$.
(Ans. $\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$)
(b) Define $T:P_2(\mathbb{R})\to M_{2\times2}$ by $T(f(x))=\begin{bmatrix}f'(0)&2f(1)\\0&f''(3)\end{bmatrix}$ where $'$ denotes differentiation. Compute $[T]_\beta^\alpha$.
(Ans. $\begin{bmatrix}0&1&0\\2&2&2\\0&0&0\\0&0&2\end{bmatrix}$)

15. (Section 2.2) Let $V$ be a vector space with the basis $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$. Define $\mathbf{v}_0=\vec{0}$. We know that there exists a unique linear transformation $T:V\to V$ such that $T(\mathbf{v}_j)=\mathbf{v}_j+\mathbf{v}_{j-1}$ for $j=1,2,\ldots,n$. Compute the matrix of $T$.

16. (Section 2.3) Let $U:\mathbb{R}^2\to\mathbb{R}^3$ be given by $U(x_1,x_2)=(x_1,0,0)$, and let $T:\mathbb{R}^2\to\mathbb{R}^2$ be given by $T(x_1,x_2)=(-x_2,0)$. Taking the standard basis of both $\mathbb{R}^2$ and $\mathbb{R}^3$ find $[U],[T]$, and $[U\circ T]$. Verify that $[U\circ T]=[U][T]$. (Discussed in class)

17. (Section 2.3) Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations defined by $T(f(x)) = f'(x)(3+x)+2f(x)$ and $U(a+bx+cx^2) = (a+b, c, a-b)$ respectively. Let $\beta$ and $\gamma$ be the standard bases of $P_2$ and $\mathbb{R}^3$ respectively. Compute the matrices of the transformations $T$, $U$ and $U \circ T$, i.e., compute $[T]_\beta^\beta$, $[U]_\beta^\gamma$ and $[U \circ T]_\beta^\gamma$ directly. Then verify that $[U \circ T]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta^\beta$.

18. (Section 2.4) Let

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Is $V$ isomorphic to $\mathbb{R}^3$? Why, or why not? If yes, then construct an isomorphism from $V$ to $\mathbb{R}^3$.
(Ans. Yes, $V$ is isomorphic to $\mathbb{R}^3$. Let $\phi : V \to \mathbb{R}^3$ be defined as $\phi(\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}) = (\alpha, \beta - \alpha, \gamma)$; $\phi$ is an isomorphism from $V$ to $\mathbb{R}^3$.)

19. Which of the following pairs of vector spaces are isomorphic? Justify your answers.

- $\mathbb{R}^4$ and $P_3(\mathbb{R})$. (Isomorphic))
- $V = \{A \in M_{2\times 2(\mathbb{R})} : trace(A) = 0\}$ and $\mathbb{R}^4$. (Not isomorphic)

(Hint: Find the dimension of the vectors spaces in each case and argue by means of the theorem that says that being isomorphic is equivalent to having the same dimension.)

20. (Section 2.5) For the following pair of ordered bases $\beta$ and $\beta'$, and the given vector spaces $V$, find the change of coordinate matrix that changes $\beta'$ coordinates to $\beta$ corodinates.

(a) $V = \mathbb{R}^2$; $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

(b) $V = P_2(\mathbb{R})$; $\beta = \{x^2 - x, x^2 + 1, x - 1\}$ and $\beta' = \{5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3\}$
(Ans. $\begin{bmatrix} 5 & -6 & 3 \\ 0 & 4 & -1 \\ 3 & -1 & 2 \end{bmatrix}$)

21. (Section 2.5) Let $T$ be the linear transformation on $P_1(\mathbb{R})$ defined $T(p(x)) = p'(x)$. Let $\beta_1 = \{1, x\}$ and $\beta_2 = \{1 + x, 1 - x\}$ be two bases for $P_1(\mathbb{R})$. Find $[T]_{\beta_1}$ and then use this and the fact that

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

to find the matrix of $T$ with respect to $\beta_2$.