Additional Practice Problems for Exam II

MATH 430

1. (Section 5.2) For each of the following matrices A, determine if A is diagonalizable. If diagonalizable, find an invertible matrix Q such that $Q^{-1}AQ = D$ and determine A^n for some arbitrary natural number n.

   (a) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$ (Solution key: not diagonalizable, $\lambda = 3$ has only one l.i. eigenvector)

   (b) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ (Solution key: has two distinct eigenvalues)

   NOTE: In the above question you should not compute Q^{-1} and multiply the product to verify that $Q^{-1}AQ$ is diagonal. If diagonalizable, this will be true by the theory. You just have to give the matrix Q and the corresponding diagonal matrix D.

2. (Section 5.4) Consider the matrix

   $$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

   Use the Cayley Hamilton theorem to determine a) $A^4 - 2A^3 + 5A^2$, (b) $A^6 - 2A^5 + 5A^4 + 6A + I$
   (Solution key: (a) the 0 matrix, (b) find $6A + I$)

3. (Section 6.2) Applying the Gram-Schmidt procedure find an orthonormal basis for the space $H = \text{span}\{e^{-t/2}, te^{-t/2}, t^2e^{-t/2}\}$ using the inner product as $\langle f, g\rangle = \int_0^\infty f(t)g(t)dt$.
   (You may use the following:
   $\int_0^\infty e^{-t} = 1, \int_0^\infty te^{-t} = 1, \int_0^\infty t^2e^{-t} = 2, \int_0^\infty t^3e^{-t} = 6, \int_0^\infty t^4e^{-t} = 24$.)
   (Ans. $\{e^{-t/2}, (t-1)e^{-t/2}, \frac{t^2-4t+2}{2}e^{-t/2}\}$)

4. (Section 6.2) In each of the following find the orthogonal projection of the given vector on the given subspace W of the inner product space V.
   (a) $V = \mathbb{R}^3$, $u = (2, 1, 3)$, and $W = \{(x, y, z) : x + 3y - 2z = 0\}$
   (b) $V = P(\mathbb{R})$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$, $h(x) = 4 + 3x - 2x^2$, and $W = P_1(\mathbb{R})$. (Ans. : $x + \frac{13}{3}$)

5. (Section 6.3) Let x be the length of a spring and let y be the force applied to (exerted by) the spring. By Hooke's law, there is a linear relationship between the length x and the force y, i.e., $y = kx + d$, where k is called the spring constant. Use the following data to estimate the spring constant (x is in inches and y is in pounds).

Length (x)	3.5	4.0	4.5	5.0
Force (y)	1.0	2.2	2.8	4.3

(Ans. ~ 2.1) Note: In the test, you may be asked to give a clear set-up of the solution but need not simplify the final answer. In particular, you won't need to calculate any matrix inverses.

6. (Section 6.4) For the given linear operator T defined on the inner product space V, determine whether T is normal, self adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T and list the corresponding eigenvalues.
$V = M_{2\times 2}(\mathbb{R})$ and T is defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.
(Ans. key: T is self-adjoint (so there exists an orthonormal set of eigenvectors), eigenvalues: $1, 1, -1, -1$)

7. (Section 6.4) For the given linear operator T and the given vector space V determine whether T is normal or self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.
(a) $V = \mathbb{R}^3$ and T is defined by $T(a, b, c) = (-a + b, 5b, 4a - 2b + 5c)$.
(b) $V = M_{2\times 2}(\mathbb{R})$ and T is defined by $T(A) = A^t$ where A^t is the transpose of A.
(In each case you can use the standard basis for the space V.)

8. (Section 6.5) For the following matrices A find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$.

$$\text{(a) } A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\text{Ans. } P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\text{(b) } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

$$\text{Ans. } P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

9. State whether the following are true or false (with justification):
   (a) A matrix $M \in M_{n\times n}$ has rank n if and only if $\det(M) = 0$.
   (b) For $A \in M_{n\times n}$, $\det(A^t) = -\det(A)$, where $\det(A^t)$ is the transpose of A.

(c) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

(d) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(e) The adjoint of an unitary operator is unitary.

(f) Every real symmetric matrix is diagonalizable.

(g) Every self-adjoint operator is normal.

(h) Operators and their adjoints have the same eigenvectors.