Homework 7

MATH 430

All work must be shown clearly for full credit. You must justify all your answers.
Points will be deducted for incomplete/incorrect/haphazard/unorganized work.

Section 6.1

1. In $C([0,1])$ (the space of continuous functions defined on $[0,1]$), define inner product by $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$. Let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g\rangle$, $\|f\|$, $\|g\|$ and $\|f+g\|$. Verify both the Cauchy-Schwarz and the triangle inequality.

2. (Pythagorean Theorem) Let V be an inner product space, and suppose that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors in V. Prove that $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

3. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k$ be scalars. Prove that

$$\left\|\sum_{i=1}^{k} a_i\mathbf{v}_i\right\|^2 = \sum_{i=1}^{k} |a_i|^2\|\mathbf{v}_i\|^2.$$

Section 6.2

4. In the following, apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthonormal basis β for span(S). Then express the given vector h as a linear combination of the vectors in β.

   (a) $V = \mathbb{R}^3$, $S = \{(1,0,1), (0,1,1), (1,3,3)\}$, and $h = (1,1,2)$.
   (The inner product for $\mathbb{R}^3$ is the usual dot product.)

   (b) $V = P_2(\mathbb{R})$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$. $S = \{1, x, x^2\}$ and $h = 1 + x$.