Additional practice problems

MATH 472

1. Is $\cap_{n=1}^{\infty}[-1, \frac{1}{n})$ open, closed, neither?

2. (a) Give an example to show that the arbitrary intersection of open sets need not be open.
   (b) Give an example to show that the arbitrary union of closed sets need not be closed
   (For the sake of some exercise, you should try to come up with example different from the ones discussed in the lecture.)

3. Find the limit if it exists of the sequence $\{(\frac{1}{n}, n^{-n})\}$ in $\mathbb{R}^2$.

4. Show that the following limits do not exist:
   (a) $\lim_{(x,y)\to(0,0)} \frac{x+y^2}{y^2}$
   (b) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ (Take $y = mx$; the limit depends on m)

5. Show that the function $f(x,y) = \begin{cases} \frac{x-y}{x+y} & \text{if } y \neq -x \\ 1 & \text{if } y = -x \end{cases}$ is not continuous.

6. (a) Determine whether $f(x,y) = \cos(x+y)+x^2y^2$ is continuous. Justify your answer.
   (b) Using the ϵ-δ criteria show that
   $$f(x,y) = \begin{cases} \frac{3x^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$
   is continuous at the origin.

7. Show that if $U \subset$ is an open set, then $A = \{(x,y) \in \mathbb{R}^2 \ : \ x \in U\}$ is open.
   (Hint: Use the fact that the inverse image of an open set under a continuous function is open and then work with an appropriate continuous function.)

8. Define
   $$g(x,y) = \begin{cases} \frac{x^2y^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$
   Prove that g has first order partial derivatives. Is g continuously differentiable?

9. Find the directional derivative of $f(x, y) = x + \sin y^2$ at $(1, 0)$ in the direction of $(1, -2)$.

10. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* if $f_{xx} + f_{yy} = 0$ for all $(x, y) \in \mathbb{R}^2$. Determine whether the given functions are harmonic or not.
    (a) $f(x, y) = e^x \cos y$
    (b) $f(x, y) = x^2 - x^3$

11. Find the unit vector in the direction in which $f(x, y) = y^2 \sin x$ changes most rapidly at the point $(0, -2)$. What is the maximum rate of change of f at $(0, -2)$?