MATH 472: Practice Problems

Problems on line integrals, conservative vector fields, Green's Theorem

1. Evaluate the following line integrals
   (a) $\int_C (x+\sqrt{y})\ \mathrm{d}s$, where C is parametrized by $\gamma(t) = (2t, t^2)$, for $0 \le t \le 1$.
   (b) $\int_C xy\ \mathrm{d}s$, where C is the parabola $y = x^2$ between the points $(0,0)$ and $(2,4)$.
   (c) $\int_C x^3y^2\ \mathrm{d}s$, where C is the quarter of the unit circle in the first quadrant.

2. Find the work done by the force field $\overrightarrow{F}(x,y) = (x^2y, xy^2)$ in moving an object along the line segment from the point $(0,0)$ to the point $(2,3)$.

3. Determine whether or not the given vector field is conservative in the region D. If it is, find the potential function f.
   (a) $\overrightarrow{F}(x,y) = (y^2, 2xy)$, $D = \mathbb{R}^2$.
   (b) $\overrightarrow{F}(x,y) = (3y + 4xy^2, 3x + 3x^2y)$, $D = \mathbb{R}^2$.

4. Use Green's Theorem to calculate the area of the astroid given by $\overrightarrow{r}(t) = (\cos^3 t, \sin^3 t)$ with $t \in [0, 2\pi]$.

5. (Change of variables) If D is the trapezoid with vertices $(0,1), (0,2), (2,0)$, and $(1,0)$, find
$$\iint_D \sin\frac{y-x}{y+x}.$$

6. Suppose that C is a positively oriented boundary of an annular region between the circles $x^2+y^2 = 4$ and $x^2+y^2 = 16$. If $\overrightarrow{F}(x,y) = (-xy, x)$ is a vector field, calculate the line integral of $\overrightarrow{F}$ on C.
   (Try to use Green's Theorem)

7. Determine whether the given vector field is conservative and if so find its potential:
$$\overrightarrow{F}(x,y) = (3x^2y - \cos x + 4, x^3 - e^y)$$
   (Ans. conservative, $f(x,y) = x^3y - e^y + 4x - \sin x$)

8. Compute $\int_C \overrightarrow{F}.d\overrightarrow{r}$ where $\overrightarrow{F}(x,y) = (x, x+y)$ and C is the unit circle $x^2 + y^2 = 1$ traced in the counterclockwise direction. (Try this both ways: using and not using Green's Theorem.)
(Ans. π)

Additional problems

1. Prove that the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$ diverges to ∞.
(Consider $\lim_{k\to\infty} \frac{k}{k+1}$.)

2. Determine whether the given series converges or diverges
(a) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ (Converges to 1. Find the partial sum s_n and take its limit.)
(b) $\sum_{k=1}^{\infty} \ln \frac{1}{k}$ (Diverges)

3. Find the sum of $3 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$
(Ans. 5)

4. Test the convergence of the following series
(a) $\sum_{k=1}^{\infty} \frac{5}{3^k+2}$ (Converges. Compare with $\sum \frac{1}{3^k}$)
(b) $\sum_{k=1}^{\infty} \frac{1}{2+\sqrt{k}}$ (Diverges. Compare with $\sum \frac{1}{k}$)
(c) $\sum_{k=1}^{\infty} ke^{-k^2}$ (Converegs. Integral Test.)

5. Determine whether the following series converges conditionally, converges absolutely or diverges
(a) $\sum_k \frac{k}{(-2)^{k-1}}$ (Ans. Converges by the alternating series test, converges absolutely by the ratio test)
(b) $\sum_k \frac{(-1)^k}{k \ln k}$ (Ans. Converges conditionally)

6. Find the pointwise limit, if it exists, of the given sequence $\{f_n\}$
(a) $f_n : [-1, 1] \to \mathbb{R}$; $f_n(x) = \frac{nx}{1+n^2x^2}$
(Ans. $f(x) = 0$)
(b) $f_n : [0, \infty) \to \mathbb{R}$; $f_n(x) = \frac{x^n}{1+x^{2n}}$
(Ans. $f(1) = \frac{1}{2}$, $f(x) = 0$ when $x \neq 1$.)
(c) $f_n : [0, 1] \to \mathbb{R}$; $f_n(x) = nxe^{-nx^2}$
(Ans.$f(x) = 0$)

7. Prove that for $f_n : [0, \infty) \to \mathbb{R}$
(a) $f_n(x) = x^n e^{-nx}$ converges uniformly.
(b) $f_n = \frac{nx}{1+n^2x^2}$ converges only pointwise

8. Suppose that $\{f_n\}$ with $f_n : [2,5] \to \mathbb{R}$ is defined by $f_n(x) = \frac{x^n}{1+x^{2n}}$. Find $\lim_{n\to\infty} \int_2^5 f_n(x)\, \mathrm{d}x$.

9. Prove that $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is continuous on $[0,1]$. (Use the Weierstrass M-Test to first show uniform convergence of the series.)

10. For a natural number n and reals $a_1, \ldots, a_n$, verify that

$$|a_1 + \cdots + a_n| \leq \sqrt{n}\sqrt{a_1^2 + \cdots + a_n^2}.$$

(Use Cauchy-Schwarz Inequality)

11. Relevant problems given in HWs 1 - 6, problems discussed in the lectures, problems handed out for practice before Midterms I & II