Practice Problems for Midterm II

MATH 420

1. Evaluate
$$\int_C \frac{z+2}{3z},$$
C is the sector of the circle $|z| = 3$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$ i.e. counter-clockwise.

2. Evaluate the integrals of the following functions about the given curve C:

   (a) $f(z) = \frac{z^2}{z-3}$; $C =$ the unit circle$|z| = 1$
   (Ans. $\int_C f(z) = 0$ by Cauchy's Integral Theorem since f is analytic everywhere on and within C.)

   (b) $f(z) = ze^{-z}$; $C =$ the unit circle$|z| = 1$
   (Ans. $\int_C f(z) = 0$ by Cauchy's Integral Theorem since f is analytic everywhere on and within C.)

3. Let C denote the positively oriented (traveling counter-clockwise) square whose sides lie along the line $x = \pm 2$ and $y = \pm 2$. Evaluate the integrals:
(a) $\int_C \frac{e^{-z}\mathrm{d}z}{z-(\frac{\pi i}{2})}$ (Ans. 2π)
(b) $\int_C \frac{\cos z}{z(z^2+8)}\, \mathrm{d}z$ (Ans. $\pi i/4$)

4. Let C denote the circle $|z - i| = 2$ positively oriented. Evaluate $\int_C \frac{1}{z^2+4}\mathrm{d}z$
(Ans. $\pi/2$)

5. Evaluate $\int_C \frac{e^{-z}}{z^2}\, \mathrm{d}z$; C is the unit circle $|z| = 1$ (Ans. $-2\pi i$)

6. Find a series expansion about $z = 0$ of $\frac{1}{1-z^4}$; find the region of convergence or the region where the series is valid.

7. Find a series expansion about $z = 1$ of $\frac{1}{z^2-1}$; find the region where this is valid.

8. (**Singular points**) Find all the singularities of the given functions and classify them:
(a) $\frac{e^z-1}{z}$
(b) $ze^{\frac{1}{z}}$ (Ans. $z = 0$ is an essential singularity)
(c) $\frac{1}{z^2}\cos 2z$
(d) $\frac{1}{z^2(z-3)^2}$ (Ans. $z = 0$ and $z = 3$, both are poles of order 2)

Useful series expansions:
(a) $e^z = 1 + z + \frac{z}{2!} + \frac{z^3}{3!} + \cdots$
(b) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$
(c) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$
(d) $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots;$ for $|z| < 1$